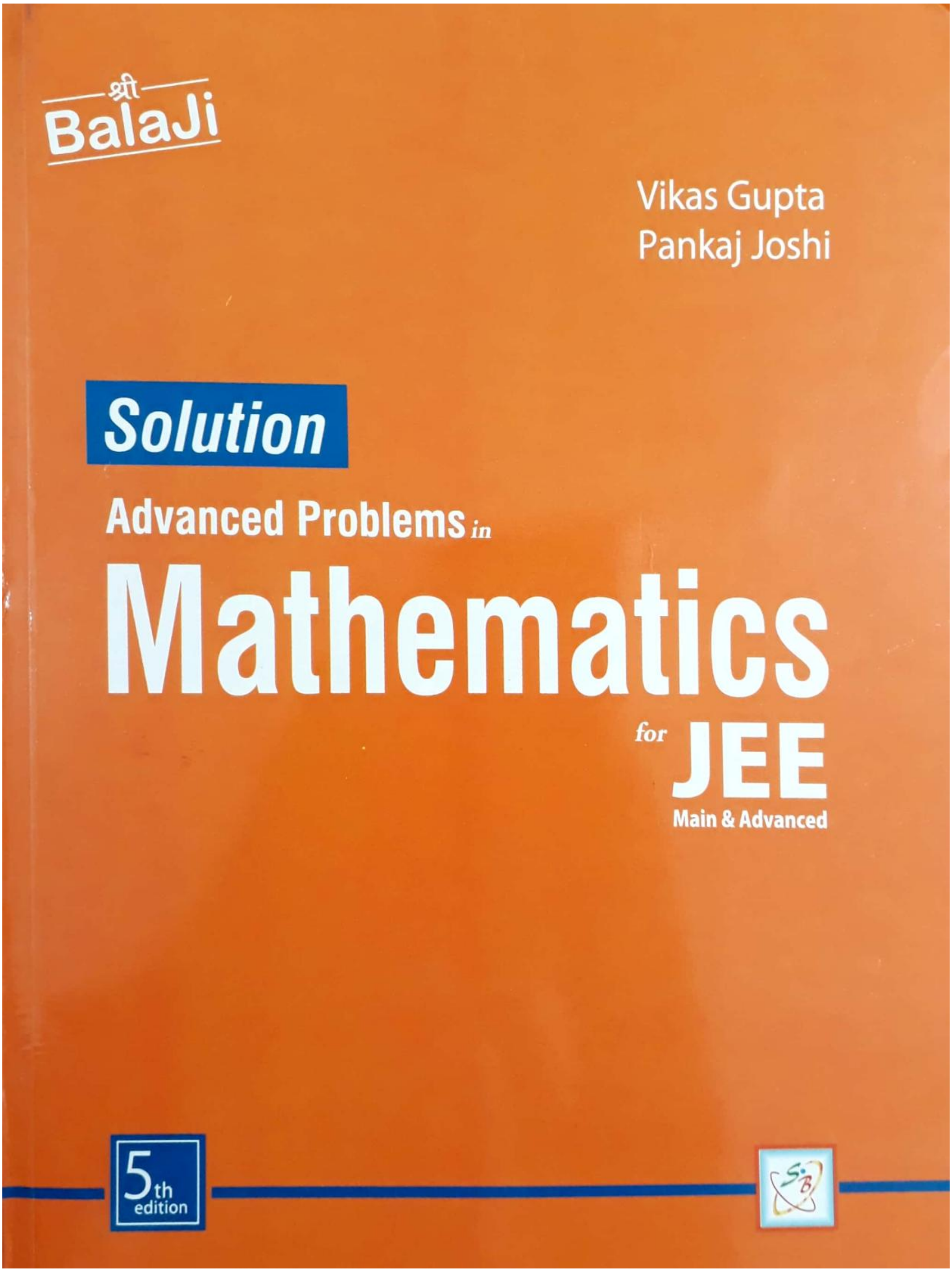
Balaji

Solution to Advanced Problems in Mathematics Chapter 9 to 26

for IIT JEE Main and Advanced

by

Vikas Gupta and Pankaj Joshi



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BalaJi

Vikas Gupta
Pankaj Joshi

Solution

Advanced Problems in Mathematics for JEE

Main & Advanced

5th edition

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SOLUTION *to*

Advanced Problems
in
MATHEMATICS
for
JEE (MAIN & ADVANCED)

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CONTENTS

CALCULUS

1. Function 1 – 28
2. Limit 29 – 42
3. Continuity, Differentiability and Differentiation 43 – 73
4. Application of Derivatives 74 – 101
5. Indefinite and Definite Integration 102 – 134
6. Area Under Curves 135 – 141
7. Differential Equations 142 – 149

ALGEBRA

8. Quadratic Equations 150 – 186
9. Sequence and Series 187 – 207
10. Determinants 208 – 215
11. Complex Numbers 216 – 223
12. Matrices 224 – 228
13. Permutation and Combinations 229 – 235
14. Binomial Theorem 236 – 241
15. Probability 242 – 247
16. Logarithms 248 – 261

CO-ORDINATE GEOMETRY

17. Straight Lines 262 – 273
18. Circle 274 – 284
19. Parabola 285 – 289
20. Ellipse 290 – 293
21. Hyperbola 294 – 295

TRIGONOMETRY

22. Compound Angles 296 – 318
23. Trigonometric Equations 319 – 327
24. Solution of Triangles 328 – 342
25. Inverse Trigonometric Functions 343 – 351

VECTOR & 3DIMENSIONAL GEOMETRY

26. Vector & 3Dimensional Geometry 352 – 368

Chapter 9 – Sequence and Series

9

SEQUENCE AND SERIES

Exercise-1 : Single Choice Problems

1. AM $\geq$ GM

3. $2\sec\alpha = \sec(\alpha - 2\beta) + \sec(\alpha + 2\beta)$

$$\frac{2}{\cos\alpha} = \frac{\cos(\alpha + 2\beta) + \cos(\alpha - 2\beta)}{\cos(\alpha - 2\beta)\cos(\alpha + 2\beta)}$$

$$\cos 2\alpha + \cos 4\beta = \cos\alpha\,(2\cos\alpha\cos 2\beta)$$

$$2\cos^2\alpha - 1 + 2\cos^2 2\beta - 1 = 2\cos^2\alpha\cos 2\beta$$

$$\Rightarrow \quad \cos^2\alpha(1 - \cos 2\beta) + (\cos 2\beta + 1)(\cos 2\beta - 1) = 0$$

$$\Rightarrow \quad \cos^2\alpha = 1 + \cos 2\beta$$

4. If a, b, c A.P. $\Rightarrow$ $b = \dfrac{a + c}{2}$

if c, d, e H.P. $\Rightarrow$ $d = \dfrac{2ec}{e + c}$

if b, c, d G.P. $\Rightarrow$ $c^2 = bd$

$$c^2 = \left(\frac{a + c}{2}\right)\left(\frac{2ec}{e + c}\right)$$

$$\Rightarrow \quad c^2 = ae$$

5. $(a + nd)^2 = (a + md)(a + rd)$

$$\Rightarrow \quad \frac{a}{d} = \frac{mr - n^2}{2n - m - r}$$

if m, n, r in H.P., then $n = \dfrac{2mr}{m + r}$ $\Rightarrow$ $\dfrac{a}{d} = \dfrac{-n}{2}$

6. A.M. $(\alpha, \beta, \gamma, \delta) = \dfrac{4}{4} = 1$

G.M. $(\alpha, \beta, \gamma, \delta) = 1$ $\Rightarrow$ $\alpha = \beta = \gamma = \delta = 1$

So, equation is $(x - 1)^4 = 0$

7. $S_3 = S_1^2 \Rightarrow \dfrac{S_1^4S_2^2 - S_2^2S_3^2}{S_1^2 + S_3^2} = \dfrac{S_2^2(S_1^4 - S_3^2)}{S_1^2 + S_3^2} = 0$

8. $T_r = \dfrac{r \cdot 2^r}{(r+2)!}$

$$T_r = \frac{(r+2-2)\,2^r}{(r+2)!} = \frac{1}{(r+1)!}2^r - \frac{1}{(r+2)!}2^{r+1}$$

$$S_n = \frac{2!}{2!} - \frac{2^{n+1}}{(n+2)!}$$

$$\lim_{n\to\infty} S_n = S_\infty = 1 \qquad \left[\text{as } \lim_{n\to\infty} \frac{2^{n+1}}{(n+2)!} = 0\right]$$

9. $\tan^2\dfrac{\pi}{12} = \tan\left(\dfrac{\pi}{12} - x\right)\tan\left(\dfrac{\pi}{12} + x\right)$

$$\tan^2\frac{\pi}{12} = \frac{\tan^2\frac{\pi}{12} - \tan^2 x}{1 - \tan^2\frac{\pi}{12}\tan^2 x} \Rightarrow \tan^2 x\left(\tan^4\frac{\pi}{12} - 1\right) = 0 \Rightarrow \tan x = 0$$

$x = 0, \pi, 2\pi, 3\pi, \ldots\ldots 99\pi$

10. $\dfrac{S_n}{S_n - 1} = \dfrac{n}{n-1}\dfrac{n+1}{n+2}$

$$Q_n = \left(\frac{2}{1} \times \frac{3}{2} \times \frac{4}{3} \times \frac{5}{4} \times \ldots\ldots \frac{n}{n-1}\right) \times \left(\frac{3}{4} \times \frac{4}{5} \times \frac{5}{6} \times \ldots\ldots \times \frac{n}{n+1} \times \frac{n+1}{n+2}\right)$$

$$Q_n = \left(\frac{n}{1}\right) \cdot \left(\frac{3}{n+2}\right) = \frac{3n}{n+2}$$

$$\lim_{n\to\infty} Q_n = 3$$

11. $\begin{vmatrix} \log l & p & 1 \\ \log m & q & 1 \\ \log n & r & 1 \end{vmatrix} = \begin{vmatrix} \log A + (p-1)\log R & p & 1 \\ \log A + (q-1)\log R & q & 1 \\ \log A + (r-1)\log R & r & 1 \end{vmatrix}$

$$\begin{vmatrix} \log A & p & 1 \\ \log A & q & 1 \\ \log A & r & 1 \end{vmatrix} + \begin{vmatrix} (p-1)\log R & p & 1 \\ (q-1)\log R & q & 1 \\ (r-1)\log R & r & 1 \end{vmatrix} = 0$$

12. Numbers divisible by 6 → 49

Numbers divisible by 18 → 16

13. $\dfrac{y+z}{2} = \sqrt{yz} \qquad \Rightarrow \quad 1 - x \geq 2\sqrt{yz}$

Thus, $(1-x)(1-y)(1-z)\geq 2\sqrt{yz}\cdot 2\sqrt{zx}\cdot 2\sqrt{xy}=8xyz$

$\Rightarrow$ $\dfrac{xyz}{(1-x)(1-y)(1-z)}\leq\dfrac{1}{8}$

17. Clearly, both roots are lies in between −1 and 1.

$\therefore$ $$\lim_{n\to\infty}\sum_{r=1}^{n}(\alpha^r+\beta^r)=\left(\lim_{n\to\infty}\sum_{r=1}^{n}\alpha^r\right)+\left(\lim_{n\to\infty}\sum_{r=1}^{n}\beta^r\right)$$

$$=\frac{\alpha}{1-\alpha}+\frac{\beta}{1-\beta}=\frac{1}{12}$$

18. $\sum\dfrac{a_i}{a_j}=\dfrac{a_1}{a_2}+\dfrac{a_1}{a_3}+\dfrac{a_1}{a_4}+\dfrac{a_2}{a_1}+\dfrac{a_2}{a_3}+\dfrac{a_2}{a_4}+\dfrac{a_3}{a_1}+\dfrac{a_3}{a_2}+\dfrac{a_3}{a_4}+\dfrac{a_4}{a_1}+\dfrac{a_4}{a_2}+\dfrac{a_4}{a_3}$

≥ 12 $\left(\because x+\dfrac{1}{x}\geq 2\right)$

19. $\dfrac{x^2+2xy+2xy+4y^2+z^2+z^2}{6}\geq\sqrt[6]{2^2\cdot 4\cdot x^4\cdot y^4\cdot z^4}$

20. Let first term be '*a*' and difference be *d*.

$\Rightarrow$ $5(a+4d)=8(a+7d)$

$\Rightarrow$ $a+12d=0$

$$S_{25}=\frac{25}{2}[2a+24d]$$

$$S_{25}=25(a+12d)=0$$

21. $10\sin x=\sqrt{5}(4\sin^2 x+1)$ $\sin x\neq 0$

$\Rightarrow$ $\sin x=\dfrac{\sqrt{5}\pm 1}{4}$

22. Let first term of G.P. be *a* and ratio be *r*.

$\Rightarrow$ $a+ar+ar^2=70$ and $10ar=4a+4ar^2$

$\Rightarrow$ $a=40$ $r=\dfrac{1}{2}$

$$S=\frac{a}{1-r}=\frac{40}{1-\frac{1}{2}}=80$$

23. $\displaystyle\sum_{n=1}^{\infty}\frac{k}{2^{n+k}}=\frac{k}{2^k}\sum_{n=1}^{\infty}\frac{1}{2^n}=\frac{k}{2^k}$

$\displaystyle\sum_{k=1}^{\infty}\frac{k}{2^k}=\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\frac{4}{2^4}+\ldots\ldots=2$

24. $(pqr)^{1/3}\geq\dfrac{p+q+r}{3}$ $\Rightarrow p=q=r$

if $3p+4q+5r=12$ $\Rightarrow p=q=r=1$

25. $$\frac{1}{3}\left(\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}+\ldots\ldots\right)=\frac{1}{3}\left[\frac{1}{3}\left(1+\frac{1}{2}\right)+\frac{1}{5}\left(\frac{1}{2}+\frac{1}{3}\right)+\frac{1}{7}\left(\frac{1}{3}+\frac{1}{4}\right)+\ldots\ldots\right]$$

$$=\frac{1}{3}\left[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots\ldots\right]=\frac{1}{3}$$

26. $$\frac{\frac{a}{2}[2A+(a-1)D]}{a^2}=\frac{\frac{b}{2}[2A+(b-1)D]}{b^2}=c \Rightarrow D=2c, A=c$$

27. $$\frac{x/r}{1-r}=4 \Rightarrow \frac{x}{4}=r-r^2$$

if $-1<r<1$ then $-2<r-r^2<\frac{1}{4}$

$$-2<\frac{x}{4}<\frac{1}{4} \Rightarrow -8<x<1$$

28. $$t_1+t_3+t_5+\ldots\ldots+t_{2n+1}=\frac{n+1}{2}[2a+n(2d)]=248$$

$$t_2+t_4+t_6+\ldots\ldots+t_{2n}=\frac{n}{2}[2(a+d)+(n-1)2d]=217$$

$$t_{2n+1}-t_1=2n\cdot d=56$$

$$\Rightarrow \quad \frac{n+1}{2}[2a+56]=248 \text{ and } \frac{n}{2}[2a+56]=217$$

$$\Rightarrow \quad n=7,\ a=3$$

29. length of side $A_1=20$

length of side $A_2=\frac{20}{\sqrt{2}}$

length of side $A_3=\frac{20}{(\sqrt{2})^2}$

length of side $A_n=\frac{20}{(\sqrt{2})^{n-1}}$

Area of $A_n=\frac{400}{2^{n-1}}<1$

30. $$S_k=\sum_{i=0}^{\infty}\frac{1}{(k+1)^i}=1+\frac{1}{k+1}+\frac{1}{(k+1)^2}+\ldots\ldots=\frac{k+1}{k}$$

$$\sum_{k=1}^{n}kS_k=\sum_{k=1}^{n}(k+1)=\sum_{k=1}^{n}k+\sum_{k=1}^{n}1=\frac{n(n+1)}{2}+n=\frac{n(n+3)}{2}$$

31. $$T_r=\frac{(r^2+1)}{r(r+1)}\cdot 2^{r-1}=\left(1+\frac{1}{r}-\frac{2}{r+1}\right)2^{r-1}$$

$$S_n = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} 2^{r-1} + \sum_{r=1}^{n}\left(\frac{2^{r-1}}{r} - \frac{2^r}{r+1}\right) = (2^{n-1}) + \left[1 - \frac{2^n}{n+1}\right] = \left(\frac{n}{n+1}\right)2^n$$

32. $\sum_{n=2}^{29}(1.5)^n = (1.5)^2 + (1.5)^3 + \ldots\ldots + (1.5)^{29}$

$$= (1.5)^2\left[\frac{(1.5)^{28} - 1}{0.5}\right] = 2k - 2(1.5)^2$$

33. $7, A_1, A_2, A_3, \ldots\ldots, A_n, 49$ are in A.P.

$$A_1 + A_2 + A_3 + \ldots\ldots + A_n = \left(\frac{n+2}{2}\right)(7+49) - (7+49)$$

$$\Rightarrow \quad \frac{n}{2} \times 56 = 364 \Rightarrow n = 13$$

34. $\frac{2}{r^2}, \frac{2}{r}, 2, 2r, 2r^2$

35. $S_n = 5n^2 + 4n$

$t_n = S_n - S_{n-1} = 10n - 1$

36. $x^3 + y^3 = (x+y)(x^2 + y^2 - xy) = a\left[b - \left(\frac{a^2 - b}{2}\right)\right] \qquad \left(\because xy = \frac{(x+y)^2 - (x^2+y^2)}{2}\right)$

$$= \frac{3ab}{2} - \frac{a^3}{2}$$

37. $S_1 = \dfrac{1}{1 - \frac{2}{3}} = 3$

$S_2 = \dfrac{3}{1 - \frac{2}{5}} = 5$

$\vdots$

$S_n = \dfrac{2n-1}{1 - \frac{2}{2n+1}} = 2n + 1$

$$\frac{1}{S_1S_2S_3} + \frac{1}{S_2S_3S_4} + \frac{1}{S_3S_4S_5} + \ldots\ldots = \frac{1}{3\cdot5\cdot7} + \frac{1}{5\cdot7\cdot9} + \frac{1}{7\cdot9\cdot11} + \ldots\ldots$$

$$S_\infty = \sum_{r=1}^{\infty} t_r = \sum_{r=1}^{\infty} \frac{1}{(2r+1)(2r+3)(2r+5)} = \sum_{r=1}^{\infty}\left[\frac{1}{(2r+1)(2r+3)} - \frac{1}{(2r+3)(2r+5)}\right]$$

38. $ar^5, 2, 5, ar^{13}$ are in G.P.

$\Rightarrow \quad (ar^9)^2 = 10$

$t_1t_2t_3\ldots\ldots t_{19} = a^{19}r^{9\times19} = (ar^9)^{19} = 10^{19/2}$

39. A.M. ≥ G.M.

$$A+\frac{1}{A}+1\geq 3;\quad B+\frac{1}{B}+1\geq 3;\quad C+\frac{1}{C}+1\geq 3;\quad D+\frac{1}{D}+1\geq 3$$

$$\left(A+\frac{1}{A}+1\right)\left(B+\frac{1}{B}+1\right)\left(C+\frac{1}{C}+1\right)\left(D+\frac{1}{D}+1\right)\geq 3^4$$

40. $(\Sigma r)^2=\Sigma r^2+2\Sigma r_1r_2$

$$\Sigma r_1r_2=\frac{a-b}{2}$$

41. $\frac{2n}{2}[2a+(2n-1)d]=x$ and $\frac{n}{2}[2(a+2nd)+(n-1)\,d]=y$

$$\Rightarrow\qquad \frac{2y}{n}-\frac{x}{n}=3nd\ \Rightarrow\ d=\frac{2y-x}{3n^2}$$

44. $2,6,2(k-1)$ are in G.P.

$\Rightarrow\ 6^2=2\times 2(k-1)$

$\Rightarrow\ k=10$

$\Rightarrow\ x^2-x-6>0$ and $|x|<100$

$\Rightarrow\ x\in(-100,-2)\cup(3,100)$

Number of integers = 193

45.
$$\sum_{r=1}^{n}\sqrt{1+T_rT_{r+1}T_{r+2}T_{r+3}}=\sum_{r=1}^{n}\sqrt{1+\left(r-\frac{3}{2}\right)\left(r-\frac{1}{2}\right)\left(r+\frac{1}{2}\right)\left(r+\frac{3}{2}\right)}$$

$$=\sum_{r=1}^{n}\sqrt{\left(r^2-\frac{5}{4}\right)^2}=\sum_{r=1}^{n}\left|r^2-\frac{5}{4}\right|$$

$$=\sum_{r=1}^{2}\left|r^2-\frac{5}{4}\right|+\sum_{r=2}^{n}\left|r^2-\frac{5}{4}\right|=\frac{1}{4}+\sum_{r=2}^{n}r^2-\sum_{r=2}^{n}\frac{5}{4}$$

46. $T_r=\sum T_r-\sum T_{r-1}=r^2+r$

$$\sum_{r=1}^{n}\frac{2008}{T_r}=2008\sum_{r=1}^{n}\frac{1}{r(r+1)}=(2008)\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+1}\right)=(2008)\frac{n}{n+1}$$

$$\lim_{h\to\infty}\frac{(2008)n}{n+1}=2008$$

48. $P(x)=\sum_{r=1}^{n}\left(x-\frac{1}{r}\right)\left(x-\frac{1}{r+1}\right)\left(x-\frac{1}{r+2}\right)$

$$\text{Absolute term}=-\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}=-\frac{1}{2}\left[\sum_{r=1}^{n}\frac{1}{r(r+1)}-\frac{1}{(r+1)(r+2)}\right]$$

$$= -\frac{1}{2}\left[\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right]$$

$$\lim_{n\to\infty} -\frac{1}{2}\left[\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right] = -\frac{1}{4}$$

50. $\frac{1}{T_1}, \frac{1}{T_2}, \frac{1}{T_3}, \ldots\ldots, \frac{1}{T_k}$ are in A.P.

$$\frac{T_2}{T_6} = \frac{\frac{1}{a}+5d}{\frac{1}{a}+d} = 9 \Rightarrow d = -\frac{2}{a}$$

$$\frac{T_{10}}{T_4} = \frac{\frac{1}{a}+3d}{\frac{1}{a}+9d} = \frac{5}{17}$$

52. $\left(1+\frac{1}{3^2}+\frac{1}{3^4}+\ldots\ldots\right)+\frac{2}{3}\left(1+\frac{1}{3^2}+\frac{1}{3^4}+\ldots\ldots\right)=\frac{15}{8}$

53. $(x-1)(x-2)(x-3)(x-4)\ldots\ldots(x-10)$

Coefficient of x^8 = sum of terms taken two at a time

$$= \frac{1}{2}[(1+2+3+\ldots\ldots+10)^2 - (1^2+2^2+\ldots\ldots+10^2)]$$

55.

$$\text{AM} = \text{GM}$$

$$\frac{\alpha+\beta+\gamma+\delta}{4} = (\alpha\beta\gamma\delta)^{1/4} = \frac{1}{2}$$

$$\Rightarrow \qquad \alpha = \beta = \gamma = \delta = \frac{1}{2}$$

56. Use AM ≥ GM

57. $\sum_{r=1}^{\infty}(\alpha^r+\beta^r) = (\alpha+\alpha^2+\alpha^3+\ldots)+(\beta+\beta^2+\beta^3+\ldots)$

$$= \frac{\alpha}{1-\alpha}+\frac{\beta}{1-\beta}$$

$$4x^2+2x-1=0 \begin{array}{l} \nearrow \alpha \\ \searrow \beta \end{array}$$

$$4\left(\frac{x}{1+x}\right)^2+2\left(\frac{x}{1+x}\right)-1=0 \Rightarrow 5x^2-1=0 \begin{array}{l} \nearrow \frac{\alpha}{1-\alpha} \\ \searrow \frac{\beta}{1-\beta} \end{array}$$

58. $2^2[1+2^3+3^3+4^3+\ldots+10^3]=4\left[\frac{10\times 11}{2}\right]^2=12100$

59. AM ≥ HM

$$\frac{b+\frac{a}{2}+\frac{a}{2}}{3}\geq\frac{3}{\frac{4}{a}+\frac{1}{b}}$$

60. $4^x-15=4^{2-x}\Rightarrow 4^x=16\Rightarrow x=2$

Common ratio $=\cos\left(\frac{2011\pi}{3}\right)=\cos\left(670\pi+\frac{\pi}{3}\right)=\frac{1}{2}$

61. AM ≥ GM

$$\frac{a^4+b^4+\frac{c^2}{2}+\frac{c^2}{2}}{4}\geq\left(\frac{a^4b^4c^4}{4}\right)^{1/4}$$

62. $x^2+y^2=x^2+\frac{1}{x^2}\geq 2$

63. $\frac{2}{1^3}+\frac{6}{1^3+2^3}+\frac{12}{1^3+2^3+3^3}+\frac{20}{1^3+2^3+3^3+4^3}+\ldots\infty$

$$=\frac{1\times 2}{1^3}+\frac{2\times 3}{1^3+2^3}+\frac{3\times 4}{1^3+2^3+3^3}+\ldots\infty$$

$$=\lim_{n\to\infty}\sum_1^n\frac{n(n+1)}{1^3+2^3+\ldots+n^3}=\lim_{n\to\infty}\sum_1^n\frac{n(n+1)}{\left(\frac{n(n+1)}{2}\right)^2}$$

$$=\lim_{n\to\infty}4\sum_1^n\frac{1}{n(n+1)}=4\lim_{n\to\infty}\sum_1^n\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$=4\lim_{n\to\infty}\left(\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\ldots+\frac{1}{n}-\frac{1}{n+1}\right)$$

$$=4\lim_{n\to\infty}\frac{n}{n+1}=4$$

64. $\frac{1}{(k-1)}\sum_{n=1}^{\infty}\left(\frac{1}{(n+1)(n+2)\ldots(n+k-1)}-\frac{1}{(n+2)(n+3)\ldots(n+k)}\right)=\frac{1}{(k-1)}\left(\frac{1}{2\cdot 3\cdot 4\ldots k}\right)$

65. $A-G=\frac{3}{2}$ and $G-H=\frac{6}{5}$

As we know,

$$G^2=AH\quad\Rightarrow\quad G^2\left(\frac{3}{2}+G\right)\left(G-\frac{6}{5}\right)\quad\Rightarrow G=6\text{ and }A=\frac{15}{2}$$

$$\Rightarrow \quad ab = 36 \text{ and } a + b = 15 \Rightarrow a = 12 \text{ and } b = 3$$

66. $S = \dfrac{2+5}{2^2 . 5^2} + \dfrac{5+8}{5^2 . 8^2} + \dfrac{8+11}{8^2 . 11^2} + \ldots = \dfrac{1}{3}\left(\dfrac{5^2 - 2^2}{2^2 . 5^2} + \dfrac{8^2 - 5^2}{5^2 . 8^2} + \dfrac{11^2 - 8^2}{8^2 . 11^2} + \ldots\right)$

$$= \frac{1}{3}\left(\frac{1}{2^2} - \frac{1}{5^2} + \frac{1}{5^2} - \frac{1}{8^2} + \frac{1}{8^2} - \frac{1}{11^2} + \ldots + \frac{1}{29^2} - \frac{1}{32^2}\right)$$

$$= \frac{1}{3}\left(\frac{1}{4} - \frac{1}{32^2}\right) = \frac{85}{1024}$$

67. $\displaystyle\sum_{r=1}^{10} \frac{r}{(r^2-1)^2 - r^2} = \sum_{r=1}^{10} \frac{r}{(r^2 - r - 1)(r^2 + r - 1)}$

$$= \frac{1}{2}\sum_{r=1}^{10}\left(\frac{1}{r^2 - r - 1} - \frac{1}{r^2 + r - 1}\right)$$

68. $\displaystyle\sum_{r=1}^{\infty} t_r = \sum_{r=1}^{\infty} \frac{r}{r^4 + r^2 + 1}$

$$= \sum_{r=1}^{\infty} \frac{r}{(r^2+1)^2 - r^2} = \sum_{r=1}^{\infty} \frac{r}{(r^2 - r + 1)(r^2 + r + 1)}$$

$$= \frac{1}{2}\sum_{r=1}^{\infty}\left(\frac{1}{r^2 - r + 1} - \frac{1}{r^2 + r + 1}\right)$$

69. $S_\infty = 1 + \dfrac{4}{5} + \dfrac{7}{5^2} + \dfrac{10}{5^3} + \ldots$

$$\frac{1}{5} \cdot S_\infty = \frac{1}{5} + \frac{4}{5^2} + \frac{7}{5^3} + \ldots$$

$$\Rightarrow \quad \frac{4}{5} S_\infty = 1 + \frac{3}{5} + \frac{3}{5^2} + \frac{3}{5^3} + \ldots = \frac{7}{4}$$

$$\Rightarrow \quad S_\infty = \frac{35}{16}$$

71. $x_1, x_2, x_3 \ldots x_{2n}$

$$\sum_{r=1}^{2n} (-1)^{r+1} x_r^2$$

$$x_1^2 - x_2^2 + x_3^2 + \ldots - x_{2n}^2$$

$$(x_1 - x_2)(x_1 + x_2 + x_3 + \ldots x_{2n})$$

$$-(x_2 - x_1)(x_1 + x_2 + x_3 + \ldots x_{2n})$$

$$-\frac{(x_{2n} - x_1)}{2n-1}\frac{2x}{2}[x_1 + x_{2n}]$$

$$\frac{x}{2x-1}(x_1^2 - x_{2n}^2)$$

72. $\frac{\alpha+\beta}{2}=9; \quad \sqrt{\alpha\beta}=4$

73. rms ≥ AM

$$\sqrt{\frac{p^2+q^2}{2}} \geq \frac{p+q}{2}$$

74. $150\times 9+\frac{n}{2}[300+(n-1)(-2)]=4500 \Rightarrow n=25$

Total term $=n+9=34$

75. $S_{20}=\frac{20}{2}[2(1-ad)+19d]=20$

$$\Rightarrow \qquad 19d-2ad=0$$

76. $\sum_{n=3}^{\infty}\frac{1}{(n-2)(n-1)n(n+1)(n+2)}=\frac{1}{4}\sum_{n=3}^{\infty}\left(\frac{1}{(n-2)(n-1)n(n+1)}-\frac{1}{(n-1)n(n+1)(n+2)}\right)$

78. $2^x+2^{2x+1}+\frac{5}{2^x}=2^x+2^{2x}+2^{2x}+\frac{1}{2^x}+\frac{1}{2^x}+\frac{1}{2^x}+\frac{1}{2^x}+\frac{1}{2^x}$

$$\Rightarrow \quad \frac{2^x+2^{2x+1}+(5/2^x)}{8}\geq\left(2^x\times(2^{2x})^2\times\frac{1}{(2^x)^5}\right)^{1/8}=1$$

$$\Rightarrow \quad 2^x+2^{2x+1}+\frac{5}{2^x}\geq 8$$

79. $\sum_{r=1}^{\infty}\left(\frac{(4r+5)}{r(5r+5)}\right)\cdot\frac{1}{5^r}=\sum_{r=1}^{\infty}\left(\frac{1}{r}-\frac{1}{5r+5}\right)\cdot\frac{1}{5^r}=\sum_{r=1}^{\infty}\left(\frac{1}{r\cdot 5^r}-\frac{1}{(r+1)\cdot 5^{r+1}}\right)=\frac{1}{5}$

Exercise-2 : One or More than One Answer is/are Correct

1. $a=\frac{a_1+a_n}{2}, \quad b=\sqrt{a_1a_n}, \quad c=\frac{2a_1a_n}{a_1+a_n}$

$\Rightarrow \quad a\geq b\geq c$ and $b^2=ac$

2. $D_1: b^2-4ac<0$

$D_2: c^2-4ab<0$

$D_3: a^2-4bc<0$

$D_1+D_2+D_3: a^2+b^2+c^2<4(ab+bc+ac)$

$$1<\frac{a^2+b^2+c^2}{ab+bc+ac}<4$$

3. If a, b, c are in H.P.

A.M. > H.M.

$$\Rightarrow \quad \frac{a+c}{2} > b \Rightarrow a+c > 2b$$

$$\Rightarrow \quad a-b > b-c$$

or $$\frac{1}{a-b} - \frac{1}{b-c} < 0$$

G.M. > H.M.

also $\sqrt{ac} > b$ or $ac > b^2$

4. $$T_p = a+(p-1)d = \frac{1}{q(p+q)}$$

$$T_q = a+(q-1)d = \frac{1}{p(p+q)} \Rightarrow a = d = \frac{1}{pq(p+q)}$$

5. (a) $a, H_1, H_2, H_3, \ldots\ldots, H_n, b$ are in H.P.

$\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \ldots\ldots, \frac{1}{H_n}, \frac{1}{b}$ are in A.P.

$$\Rightarrow \frac{1}{H_1} + \frac{1}{H_2} + \frac{1}{H_3} + \ldots\ldots + \frac{1}{H_n} = \frac{n}{2}\left(\frac{1}{a} + \frac{1}{b}\right)$$

(c) $a, A_1, A_2, A_3, \ldots\ldots, A_{2n}, b$ are in A.P.

$A_1 + A_{2n} = A_2 + A_{2n-1} = A_3 + A_{2n-2} = \ldots\ldots = a+b$

(d) $4g_2 + 5g_3 = 4r + 5r^2$

This is minimum at $r = -\frac{2}{5}$

6. a, b, c are in H.P.

$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

(a) $\frac{a+b+c}{a} - 2, \frac{a+b+c}{b} - 2, \frac{a+b+c}{c} - 2$ are in A.P.

(b) $\frac{a+b+c}{c} - 1, \frac{a+b+c}{b} - 1, \frac{a+b+c}{c} - 1$ are in A.P.

(c) $\frac{2}{b} = \frac{1}{a} + \frac{1}{c} \geq \frac{2}{\sqrt{ac}} \Rightarrow \sqrt{ac} \geq b$

$a^5 + c^5 \geq 2\,(ac)^{5/2} \geq b^5$

(d) $2ac = ab + bc$

7. Let the roots be a, ar, ar^2, ar^3 and ar^4.

$$\frac{a(r^5 - 1)}{r-1} = 40 \quad \ldots(1)$$

and $$\frac{1}{a}\frac{\left(\frac{1}{r^5}-1\right)}{\frac{1}{r}-1}=10 \qquad \ldots(2)$$

put $\frac{r^5-1}{r-1}=\frac{40}{a}$ in (2) we get $ar^2=\pm 2$

Now, $\delta=(ar^2)^5=(\pm 2)^5$

8. (a) $\because$ $2a_{k+1}=a_k+a_{k+2}$

$\therefore$ $f_k(-1)=0$

-1 is a root.

$\therefore$ Other is also real root.

(b) From (a) (-1) is root for any 'k' so any pair of equation has a common root.

(c) If one root is -1, other roots are $-c/a$ (form)

$\frac{a_{k+2}}{a_k}$ *i.e.*, $\frac{a_3}{a_1},\frac{a_4}{a_2},\frac{a_5}{a_3}$...... are not in A.P.

9. $b=\frac{a+c}{2}, d=\frac{2ce}{c+e}$

if $c^2=bd$, then $c^2=36$ $\qquad (\because a=2, e=18)$

10. If a, b, c are in A.P. then

$$a=b-d \text{ and } c=b+d$$

$$a+b+c=60 \Rightarrow b=20$$

If $(a-2), b, (c+3)$ are in G.P., then

$$400=(18-d)(23+d) \Rightarrow d=2,-7$$

12. $\frac{81+144a^4+16b^4+9c^4}{4}\geq 36abc$

$\Rightarrow$ A.M. = G.M.

$\Rightarrow$ $81=144a^4=16b^4=9c^4$

13. x, y, z A.P.

Let $x=y-\theta$ and $z=y+\theta$

$$\cos(y-\theta)+\cos y+\cos(y+\theta)=1=\frac{\sin\frac{3\theta}{2}}{\sin\frac{\theta}{2}}\cdot\cos(y)$$

$$\sin(y-\theta)+\sin y+\sin(y+\theta)=\frac{1}{\sqrt{2}}=\frac{\sin\frac{3\theta}{2}}{\sin\frac{\theta}{2}}\cdot\sin(y)\Rightarrow\cot y=\sqrt{2}$$

$$\frac{\sin\frac{3\theta}{2}}{\sin\frac{\theta}{2}}=\sqrt{\frac{3}{2}}=3-4\sin^2\frac{\theta}{2}\Rightarrow\cos\theta=\frac{\sqrt{3}-\sqrt{2}}{2\sqrt{2}}$$

15. $\dfrac{10^{n+1}+1}{10^{n+2}+1}<\dfrac{10^{m+1}+1}{10^{m+2}+1}$

$\Rightarrow\quad 10^{n+1}\cdot10^{m+2}+10^{n+1}+10^{m+2}+1<10^{n+2}\cdot10^{m+1}+10^{n+2}+10^{m+1}+1$

$\Rightarrow\quad 10^{m+1}<10^{n+1}$

16. $S_r=\sqrt{r+S_r}\quad\Rightarrow S_r^2-S_r=r$

17. 50, 48, 46, 44, ... A.P.

$T_n=50+(n-1)(-2)=0$

$\Rightarrow\qquad n=26$

18. $S_n=S_n\sum_{r=1}^{n}t_r=\sum_{r=1}^{n}\dfrac{2r+1}{1^2+2^2+3^2+\ldots+r^2}=\sum_{r=1}^{n}6\left(\dfrac{1}{r}-\dfrac{1}{r+1}\right)=6\left(1-\dfrac{1}{n+1}\right)$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 2

Sol. $T_1=A+B=0\qquad\Rightarrow A=-B$

$T_2=A\alpha+B\beta=1\qquad\Rightarrow A(\alpha-\beta)=1$

$T_3=A\alpha^2+B\beta^2=1\quad\Rightarrow A(\alpha^2-\beta^2)=1$

$T_4=A\alpha^3+B\beta^3=2\quad\Rightarrow A(\alpha^3-\beta^3)=2$

$\Rightarrow\quad\alpha+\beta=1$ and $\alpha\beta=-1$

Paragraph for Question Nos. 3 to 4

Sol. Set A: $5-D, 5, 5+D$ and

Set B: $5-d, 5, 5+d$

$\dfrac{p}{q}=\dfrac{25-D^2}{25-d^2}=\dfrac{7}{8}$

$\Rightarrow\qquad 25=8D^2-7d^2=d^2+16d+8\qquad(\because D=1+d)$

$\Rightarrow\qquad d=1$ and $D=2$

Set A $\{3,5,7\}$ and set B $\{4,5,6\}$

Paragraph for Question Nos. 5 to 7

5. $\dfrac{(x-3)+(y+1)+(z+5)}{3} \geq [(x-3)(y+1)(z+5)]^{1/3}$

$\Rightarrow \quad (x-3)(y+1)(z+5) \leq (21)^3$

6. Term is $6(x-3)\left(y+\dfrac{1}{2}\right)\left(z+\dfrac{5}{3}\right) \Rightarrow \dfrac{(x-3)+y+\dfrac{1}{2}+z+\dfrac{5}{3}}{3} \geq \left[(x-3)\left(y+\dfrac{1}{2}\right)\left(z+\dfrac{5}{3}\right)\right]^{1/3}$

$(x-3)\left(y+\dfrac{1}{2}\right)\left(z+\dfrac{5}{3}\right) \leq \dfrac{(355)^3}{6^3 \times 3^3}$

Maximum value $= \dfrac{(355)^3}{6^2 \times 3^3}$

7. $\dfrac{x+y+z}{3} \geq (xyz)^{1/3}$; $xyz \leq (20)^3$

Paragraph for Question Nos. 8 to 10

Sol. Let removed number are A and $A+1$.

$$\frac{n(n+1)}{2} - 2A - 1 = (n-2)\frac{105}{4}$$

$$2n^2 - 103n + 206 = 8A$$

$$\Rightarrow \qquad n = 50, A = 7$$

Paragraph for Question Nos. 11 to 13

Sol. $a_{n+1} - 1 = (a_n - 1)^2$

$a_n - 1 = (a_{n-1} - 1)^2$

$a_{n-1} - 1 = (a_{n-2} - 1)^2$

$(a_2 - 1) = (a_1 - 1)^2$

$a_1 - 1 = (a_0 - 1)^2$

$\Rightarrow \quad (a_n - 1)(a_{n-1} - 1)^2 (a_{n-2} - 1)^{2^2} \ldots\ldots (a_1 - 1)^{2^{n-1}} = (a_{n-1} - 1)^2 (a_{n-2} - 1)^2 \ldots\ldots (a_0 - 1)^{2^n}$

$\Rightarrow \quad (a_n - 1) = 3^{2^n}$

$$b_n = \frac{2 \cdot (3^{2^0} + 1)(3^2 + 1)\ldots\ldots(3^{2^{n-1}} + 1)}{(3^{2n} + 1)}$$

$$b_n = \frac{3^{2^n} - 1}{3^{2^n} + 1}$$

Paragraph for Question Nos. 14 to 15

$$f(n)=\sum_{r=2}^{n}\frac{4}{(r-1)\,r(r+1)}=2\sum_{r=2}^{n}\left(\frac{1}{(r-1)\,r}-\frac{1}{r(r+1)}\right)=2\left(\frac{1}{1\cdot 2}-\frac{1}{n(n+1)}\right);\ a=\lim_{n\to\infty}f(n)=1$$

14. $f(7)+f(8)=\dfrac{122}{63}$

15. $x^2+\dfrac{3}{2}x+t=0 \begin{matrix}\nearrow \alpha\\ \searrow \beta\end{matrix}$

Paragraph for Question Nos. 16 to 17

Sol. $\dfrac{a_1}{a_1+1}=\dfrac{a_2}{a_2+3}=\dfrac{a_3}{a_3+5}=\ldots\ldots\dfrac{a_{1005}}{a_{1005}+2009}=\dfrac{1}{k}$

$a_1=\dfrac{1}{k-1},a_2=\dfrac{3}{k-1},a_3=\dfrac{5}{k-1},\ldots\ldots a_{1005}=\dfrac{2009}{k-1}$

$a_1+a_2+a_3+\ldots\ldots+a_{1005}=\dfrac{(1005)^2}{k-1}=2010 \Rightarrow k-1=\dfrac{1005}{2}$

Exercise-4 : Matching Type Problems

1. (A) a, b, c are in A.P.

$b-a=c-b$

$b-a, c-b, a$ are in G.P.

$\dfrac{c-b}{b-a}=\dfrac{a}{c-b} \Rightarrow c-b=a \qquad (\because b-a=c-b)$

(B) a, x, b are in A.P.

$x=\dfrac{a+b}{2}$

a, y, z, b are in G.P.

$y=a^{2/3}b^{1/3},\quad z=a^{1/3}b^{2/3}$

(C) $a, b=ar, c=ar^2$

If $c>4b-3a$

$r^2-4r+3>0 \qquad (\because a>0)$

$(r-3)(r-1)>0$

(D) $7x^2-8x+9<0$

$a=7>0,\ D=64-252<0$

No solution

2. (A) $a+d=b+c=20$

(B) $2, G_1, G_2, G_3, G_4, G_5, G_6, 5$ are in G.P.

$G_1G_6 = G_2G_5 = G_3G_4 = 10$

(C) $a_4h_7 = a_1h_{10} = a_{10}h_1 = 6$

(D) $(2^x-5)^2 = 2\left(2^x - \frac{7}{2}\right) \Rightarrow (2^x-8)(2^x-4)=0 \Rightarrow x=3$

3. (A) $2\cdot 2^{x^2} = 2^x + 2^{x^3}$

Exponential series can't be in A.P.

(B) If $a_1, a_2, a_3, \ldots\ldots, a_n$ are in A.P.

$$a_2-a_1 = a_3-a_2 = a_4-a_3 = \ldots\ldots = a_n - a_{n-1} = d$$

$$S = -d\left(\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \ldots\ldots + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}\right)$$

$$= -d\left(\frac{\sqrt{a_n}-\sqrt{a_1}}{d}\right) = \sqrt{a_1} - \sqrt{a_n}$$

(C)
$$\frac{S_{2n}}{S_n} = \frac{\frac{2n}{2}[2a+(2n-1)d]}{\frac{n}{2}[2a+(n-1)d]} = 3$$

$$\Rightarrow \quad 2a = (n+1)d$$

$$\frac{S_{3n}}{2S_n} = \frac{\frac{3n}{2}[2a+(3n-1)d]}{\frac{2n}{2}[2a+(n-1)d]} = 3$$

(D)
$$t_1+t_5 = t_2+t_4 = 2t_3$$

$$\frac{4(t_1-t_2-t_4)+6t_3+t_5}{3t_1} = \frac{3t_1+(t_1+t_5)-4(t_2+t_4)+3(2t_3)}{3t_1} = 1$$

4. A → Q; B → P; C → T; D → S

5. (A) $\frac{1}{3}\log_2 x + \log_2 y = 5$ and $\frac{1}{3}\log_2 y + \log_2 x = 7$

$\Rightarrow \quad \log_2 x = 6$ and $\log_2 y = 3$

$\Rightarrow \quad x = 2^6$ and $y = 2^3$

(B) $\angle B = 60°$ and $b^2 = ac$

$$\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{1}{2} \Rightarrow a^2+c^2 = 2ac$$

$$\Rightarrow \quad a = c$$

(C) AM ≥ GM

$$\frac{\frac{b}{a}+\frac{c}{a}+\frac{a}{b}+\frac{c}{b}+\frac{a}{c}+\frac{b}{c}}{6} \geq 1$$

(D) $(b+c)^2 - a^2 = \lambda bc$

$$\Rightarrow \quad b^2 + c^2 - a^2 = (\lambda - 2)\,bc$$

$$\Rightarrow \quad \frac{b^2 + c^2 - a^2}{2bc} = \frac{\lambda - 2}{2}$$

$$-1 < \frac{\lambda - 2}{2} < 1$$

$$0 < \lambda < 4$$

6. $P(n) \cdot (f(n+2) - f(n)) = q(n)$

$$P(n) \cdot \left(\frac{1}{n+1} + \frac{1}{n+2}\right) = q(n)$$

$$P(n) \cdot (2n+3) = (n^2 + 3n + 2) \cdot q(n)$$

$$\Rightarrow \quad P(n) = n^2 + 3n + 2 \text{ and } q(n) = (2n+3)$$

Exercise-5 : Subjective Type Problems

1. If a, b, c, d are in A.P. with common difference 'k', then

$$9k^3 + (x-4)k^2 + 4k = 0$$

$$k\{9k^2 + (x-4)k + 4\} = 0$$

$$D \geq 0 \quad \Rightarrow \quad (x-4)^2 - 144 \geq 0$$

$$(x+8)(x-16) \geq 0$$

$$\Rightarrow \quad x \in (-\infty, -8] \cup [16, \infty)$$

2. $S = 1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + \ldots\ldots + n \cdot 2^n$

$$2 \cdot S = 1 \cdot 2^2 + 2 \cdot 2^3 + 3 \cdot 2^4 + \ldots\ldots + (n-1) \cdot 2^n + n \cdot 2^{n+1}$$

$$\Rightarrow \quad S = (n-1) \cdot 2^{n+1} + 2 = 2 + 2^{n+10}$$

$$\Rightarrow \quad 2(n-1) = 2^{10}$$

$$\Rightarrow \quad n = 513$$

3. $$\lim_{n \to \infty} \sum_{r=1}^{n} \frac{r+2}{2^{r+1} r(r+1)} = \sum_{r=1}^{\infty} \left[\frac{1}{r \cdot 2^r} - \frac{1}{(r+1) \cdot 2^{r+1}}\right] = \frac{1}{2}$$

4. $$\sum_{r=1}^{\infty} \frac{8r}{4r^4 + 1} = 2 \sum_{r=1}^{\infty} \left(\frac{1}{2r^2 - 2r + 1} - \frac{1}{2r^2 + 2r + 1}\right) = 2$$

5. Let three terms in A.P. $a-d, a, a+d$

If $(a-d)^2, a^2, (a+d)^2$ are in G.P. $\Rightarrow d = \pm\sqrt{2}\,a$

$$r = \frac{a^2}{(a-d)^2} = \frac{1}{(1\pm\sqrt{2})^2}$$

6. $\sqrt{\frac{10^{2n}-1}{9} - 2\left(\frac{10^n-1}{9}\right)} = P\left(\frac{10^n-1}{9}\right) \Rightarrow P = 3$

7. $a-d, a, a+d, a-d+30$

If last three terms are in G.P.

$$(a+d)^2 = a(a-d+30)$$

$$\Rightarrow \qquad a = \frac{d^2}{30-3d}$$

8. $\frac{1}{8n^4}\sum_{k=1}^{n}[k(k+2)(k+4)(k+6) - (k-2)k(k+2)(k+4)]$

$$\frac{1}{8}\left[\frac{(n-1)(n+1)(n+3)(n+5)+n(n+2)(n+4)(n+6)+15}{n^4}\right] = \frac{1}{4}(n \to \infty)$$

9. Unit digit of $\left[\frac{n(n+1)}{2}\right]^2 = 1$

Then unit digit of $\frac{n(n+1)}{2}$ is 1 because unit digit of $n(n+1)$ can not be 8.

10. $2\log_b c = \log_c a + \log_a b$

$$2\left(\frac{\log a + 2\log r}{\log a + \log r}\right) = \left(\frac{\log a}{\log a + 2\log r}\right) + \left(\frac{\log a + \log r}{\log a}\right)$$

Let $A = \log a$ and $R = \log r \Rightarrow 3A^2 + 3Ar - 2R^2 = 0 \Rightarrow \frac{A}{R} = \frac{-3+\sqrt{33}}{6}$

$$d = \log_b c - \log_c a = \frac{A+2R}{A+R} - \frac{A}{A+2R} = \frac{3}{2}$$

11. $3, \frac{3}{r}, \frac{3r}{s}, 7s;\ \frac{2}{r} = 1 + \frac{r}{s}$ and $\frac{6r}{s} = \frac{3}{r} + 7s$

$\Rightarrow\ 7r^3 - 6r^2 + 21r - 18 = 0 \Rightarrow (r^2+3)(7r-6) = 0$

$$\Rightarrow \qquad r = \frac{6}{7} \text{ and } s = \frac{9}{14}$$

12.

$$S = \frac{1^2}{3^1} + \frac{2^2}{3^2} + \frac{3^2}{3^3} + \frac{4^2}{3^4} + \ldots\ldots$$

$$\frac{S}{3} = \frac{1^2}{3^2} + \frac{2^2}{3^3} + \frac{3^2}{3^4} + \ldots\ldots$$

$$\frac{2S}{3} = S - \frac{S}{3} = \frac{1}{3} + \frac{3}{3^2} + \frac{5}{3^3} + \frac{7}{3^4} + \ldots\ldots$$

$$\frac{2S}{9} = \frac{1}{3^2} + \frac{3}{3^3} + \frac{5}{3^4} + \ldots\ldots$$

$$\frac{2S}{3} - \frac{2S}{9} = \frac{1}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \frac{2}{3^4} + \ldots\ldots$$

$$\frac{4S}{9} = \frac{1}{3} + \frac{2}{3^2}\left(1 + \frac{1}{3} + \frac{1}{3^2} + \ldots\ldots\right)$$

$$\frac{4S}{9} = \frac{1}{3} + \frac{2}{9}\left(\frac{1}{2/3}\right) = \frac{2}{3} \Rightarrow S = \frac{3}{2} = \frac{p}{q}$$

13. $S_\infty = f(x)_{\max}$ $\quad x \in [-4, 3]$

$a - ar = f'(0) = 3$

$f'(x) = 3x^2 + 3 > 0$ $\quad \therefore\ f(x)_{\max} = f(3) = 27 + 9 - 9 = 27$

$$S_\infty = 27 = \frac{a}{1-r}$$

$$a(1-r) = 3 \Rightarrow \frac{1}{1-r} = \frac{a}{3}$$

$$\therefore \quad 27 = a\left(\frac{a}{3}\right)$$

$$a^2 = 81 \Rightarrow a = \pm 9$$

If $a = 9$ $\quad 1 - r = \frac{3}{9}$ $\quad$ If $a = -9$ $\quad 1 - r = -\frac{1}{3}$

$r = \frac{2}{3}$ $\quad r = \frac{4}{3} > 1$ (rejected)

$\therefore \quad \frac{p}{q} = \frac{2}{3}$ $\quad \therefore\ p + q = 5$

14. Total runs from 1 to 9 = 1350

Let, number of terms in A.P. be n.

$\Rightarrow \quad \frac{n}{2}[300 + (n-1) \times (-1)] = 4500 - 1350 = 3150$

$\Rightarrow \quad n = 25$ or 126, $n = 126$ (not possible)

$\Rightarrow \quad n = 25$, total matches = 34

15. $x = \frac{10}{4} \sum_{n=3}^{100} \left(\frac{1}{n-2} - \frac{1}{n+2}\right) = \frac{10}{4}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{102} - \frac{1}{101} - \frac{1}{100} - \frac{1}{99}\right)$

16. $$f(n) = \frac{(2n+1)+(2n-1)+\sqrt{(2n+1)(2n-1)}}{\sqrt{2n+1}+\sqrt{2n-1}}$$

Let $\sqrt{2n+1} = a$ and $\sqrt{2n-1} = b$

$$f(n) = \frac{(a^2+b^2+ab)}{(a+b)}\frac{(a-b)}{(a-b)} = \frac{a^3-b^3}{a^2-b^2}$$

$$\Rightarrow \qquad f(n) = \frac{(2n+1)^{3/2}-(2n-1)^{3/2}}{2}$$

$$\sum_{n=1}^{60} f(n) = \sum_{n=1}^{60} \frac{(2n+1)^{3/2}-(2n-1)^{3/2}}{2} = \frac{(121)^{3/2}-1}{2} = 665$$

17. $3^0\{2^0+2^{-1}+2^{-2}\ldots\infty\} = 1\{2\}$

$3^{-1}\{2^0+2^{-1}+2^{-2}\ldots\infty\} = \frac{1}{3}\{2\}$

$3^{-2}\{2^0+2^{-1}+2^{-2}\ldots\infty\} = \frac{1}{3}\{2\}$

$\vdots \qquad\qquad\qquad\qquad \vdots$

$\qquad\qquad\qquad\qquad\qquad \infty$

Hence, $\dfrac{2\times 1}{1-\dfrac{1}{3}} = 3$

18. $15^2+(15+d)^2+(15+2d)^2+\ldots+(15+9d)^2 = 1185$

$\Rightarrow \quad 19d^2+90d+71 = 0$

$\Rightarrow \qquad d = -1$

$S_n \geq S_{n-1}$

$\frac{n}{2}(31-n) \geq \left(\frac{n-1}{2}\right)(32-n) \Rightarrow n \leq 16$

19. $$24x^3-14x^2+kx+3 = 0 \begin{array}{l} \nearrow a/r \\ \rightarrow a \\ \searrow ar \end{array}$$

Product of roots $a^3 = -\frac{1}{8} \Rightarrow a = -\frac{1}{2}$

$\Rightarrow \quad k = -7$

If $x = 7$ lies between the roots, then

$$f(7) = 49 + 7\alpha^2 - 112 < 0$$

$$\alpha^2 - 9 < 0$$

20. $9x^3+3y^3+1 = 9xy$

$(9^{1/3}x)^3+(3^{1/3}y)^3+1^3 = 3(9^{1/3}x)(3^{1/3}y) \Rightarrow 9^{1/3}x = 3^{1/3}y = 1$

21. If a, x, y, z, b A.P.

$$x = \frac{3a+b}{4}, y = \frac{a+b}{2} \text{ and } z = \frac{a+3b}{4}$$

If a, x, y, z, b H.P.

$$x = \frac{4ab}{3b+a}, y = \frac{2ab}{a+b} \text{ and } z = \frac{4ab}{3a+b}$$

If $\left(\frac{3a+b}{4}\right)\left(\frac{a+b}{2}\right)\left(\frac{a+3b}{4}\right) = 55$ and $\left(\frac{4ab}{3b+a}\right)\left(\frac{2ab}{a+b}\right)\left(\frac{4ab}{3a+b}\right) = \frac{343}{55} \Rightarrow ab = 7$

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Chapter 10 – Determinants

10

DETERMINANTS

Exercise-1 : Single Choice Problems

1. Direct expansion.

2. $$D=\begin{vmatrix} k & 1 & 1\\ 1 & k & 1\\ 1 & 1 & k\end{vmatrix}=0 \qquad \Rightarrow (k-1)^2(k+2)=0$$

also $$D_1=\begin{vmatrix} 1 & 1 & 1\\ k & k & 1\\ k^2 & 1 & k\end{vmatrix}\neq 0 \qquad \Rightarrow k\neq 1 \Rightarrow k=-2$$

3. $$\begin{vmatrix} a & a^2 & 1+a^3\\ b & b^2 & 1+b^3\\ c & c^2 & 1+c^3\end{vmatrix}=(1+abc)\begin{vmatrix} 1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}=0$$

4. $$D=\begin{vmatrix} 1 & 2a & a\\ 1 & 3b & b\\ 1 & 4c & c\end{vmatrix}=0 \quad \Rightarrow \frac{2}{b}=\frac{1}{a}+\frac{1}{c}$$

6. $$2x+ay+6z=8$$

and $4x+2ay+6z=8 \quad \Rightarrow \quad 2x+ay=0$

and $6x+12y+6z=30 \quad \Rightarrow \quad 4x+(12-a)\,y=22$

$\Rightarrow \qquad y=\dfrac{22}{12-3a} \qquad a\neq 4$

7. $R_1\rightarrow R_1+R_2+R_3$

$$\begin{vmatrix} x^2-4 & x^2-4 & x^2-4\\ 2 & x^2-13 & 2\\ x^2-13 & 3 & 7\end{vmatrix}=(x^2-4)\begin{vmatrix} 1 & 1 & 1\\ 2 & x^2-13 & 2\\ x^2-13 & 3 & 7\end{vmatrix}=0$$

$\Rightarrow \quad (x^2-4)(x^2-15)(20-x^2)=0$

8. $D = \begin{vmatrix} k & k+1 & k-1 \\ k+1 & k & k+2 \\ k-1 & k+2 & k \end{vmatrix} = 0$

$R_1 \to R_1 - R_2$ $\qquad\qquad\qquad R_2 \to R_2 - R_3$

$D = \begin{vmatrix} -1 & 1 & -3 \\ 2 & -2 & 2 \\ k-1 & k+2 & k \end{vmatrix} = 0$

9. $\Delta = \begin{vmatrix} \log a + (n-1)\log r & \log a + (n+1)\log r & \log a + (n+3)\log r \\ \log a + (n+5)\log r & \log a + (n+7)\log r & \log a + (n+9)\log r \\ \log a + (n+11)\log r & \log a + (n+13)\log r & \log a + (n+15)\log r \end{vmatrix}$

$C_3 \to C_3 - C_2$ $\qquad\qquad\qquad C_2 \to C_2 - C_1$

$\Rightarrow \begin{vmatrix} \log a + (n-1)\log r & 2\log r & 2\log r \\ \log a + (n+5)\log r & 2\log r & 2\log r \\ \log a + (n+11)\log r & 2\log r & 2\log r \end{vmatrix} = 0$

10. $D_2 = \begin{vmatrix} a_1 & 2a_3 & 5a_2 \\ b_1 & 2b_3 & 5b_2 \\ c_1 & 2c_3 & 5c_2 \end{vmatrix} = 10\begin{vmatrix} a_1 & a_3 & a_2 \\ b_1 & b_3 & b_2 \\ c_1 & c_3 & c_2 \end{vmatrix} = -10\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

11. $\Delta_2 = \begin{vmatrix} 1 & bc & a \\ 1 & ac & b \\ 1 & ab & c \end{vmatrix}$

$R_1 \to aR_1 \qquad R_2 \to bR_2 \qquad R_3 \to cR_3$

$\Delta_2 = \dfrac{1}{abc}\begin{vmatrix} a & abc & a^2 \\ b & abc & b^2 \\ c & abc & c^2 \end{vmatrix} = \begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix} = -\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = -\Delta_1$

12. $C_1 \to C_1 - C_2 + C_3 \begin{vmatrix} 0 & 0 & -1 \\ 0 & 1 & 1-a \\ 1 & a & 1+a-b \end{vmatrix} = 1$

13. $\begin{vmatrix} 1 & 2 & x \\ 2 & 3 & x^2 \\ 3 & 5 & 2 \end{vmatrix} = 10 \Rightarrow x^2 + x - 12 = 0$

Sum $= -1$

14. $R_1 \to R_1 - R_2, \quad R_2 \to R_2 - R_3$

$$D=\begin{vmatrix} -1 & -1 & -1 \\ d-a+1 & e-b+1 & f-c+1 \\ x+a & x+b & x+c \end{vmatrix},\ C_1 \to C_1 - C_2 \text{ and } C_2 \to C_2 - C_3$$

On solving D does not depend on x.

15. $R_1 \to R_1 + R_2 + R_3$

$$\Delta = (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix}\quad C_1 \to C_1 - C_2 \text{ and } C_2 \to C_2 - C_3$$

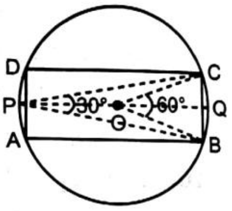
$$\Delta = (x+y+z)^3\begin{vmatrix} 0 & 0 & 1 \\ 1 & -1 & 2y \\ 0 & 1 & z-x-y \end{vmatrix} \Rightarrow \Delta = (x+y+z)^3$$

16. $\angle BOC = 60°$

$\Rightarrow\ BC = OB = OC = r$

$AB = 2r\cos 30° = \sqrt{3}\,r$

$$\frac{\text{Area or rectangle}}{\text{Area of circle}} = \frac{\sqrt{3}\,r^2}{\pi r^2} = \frac{\sqrt{3}}{\pi}$$



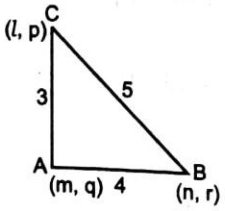
17. $C_1 \to C_1 - bC_3,\ C_2 \to C_2 + aC_3$

$$(1+a^2+b^2)^2\begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$$

18. $$\begin{vmatrix} 2 & a+b+c+d & ab+cd \\ a+b+c+d & 2(a+b)(c+d) & ab(c+d)+cd(a+b) \\ ab+cd & ab(c+d)+cd(a+b) & 2abcd \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ c+d & a+b & 0 \\ cd & ab & 0 \end{vmatrix}\begin{vmatrix} 1 & a+b & ab \\ 1 & c+d & cd \\ 0 & 0 & 0 \end{vmatrix}$$

$$=0$$

19. $$|B| = \begin{vmatrix} l & m & n \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = [2\text{Ar}(\Delta ABC)]^2$$



20. $$D = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 1 & 5 & 10 \end{vmatrix} = 0$$

$$D_1 = \begin{vmatrix} 1 & 2 & 1 \\ K & 3 & 4 \\ K^2 & 5 & 10 \end{vmatrix} = 5(K^2 - 3K + 2) = 5(K-1)(K-2)$$

$$D_2 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & K & 4 \\ 1 & K^2 & 10 \end{vmatrix} = -3(K^2 - 3K + 2) = -3(K-2)(K-1)$$

$$D_3 = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & K \\ 1 & 5 & K^2 \end{vmatrix} = K^2 - 3K + 2 = (K-2)(K-1)$$

21. $(x+1)(x+2)(x+3)\begin{vmatrix} 1 & x+1 & (x+1)^2 \\ 1 & x+2 & (x+2)^2 \\ 1 & x+3 & (x+3)^2 \end{vmatrix} = 2(x+1)(x+2)(x+3)$

22. $\begin{vmatrix} -2 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$

$-2(1-\cos^2 A) - \cos C(-\cos C - \cos A \cos B) + \cos B(\cos C \cos A + \cos B)$

$-2 + \cos 2A + \dfrac{1+\cos 2C}{2} + \dfrac{1+\cos 2B}{2} + 2\cos A \cos B \cos C$

$\cos 2A + \cos 2C + \cos 2B + 2\cos A \cos B \cos C$

$2\cos(A+B)\cos(A+B) + 2\cos^2 C - 1 + 2\cos A \cos B \cos C$

$2\cos C[\cos C - \cos(A-B)]$

$-2\cos C \cos A \cos B - 1 + 2\cos A \cos B \cos C = -1$

24. As a, b and c are the roots of $x^3 + 2x^2 + 1 = 0$, we have

$$a + b + c = -2$$
$$ab + bc + ca = 0$$
$$abc = -1$$

Now, for finding the value of $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$, evaluating using first row, we get

$$\begin{aligned} a(bc - a^2) - b(b^2 - ac) + c(ab - c^2) &= abc - a^3 - b^3 + abc + abc - c^3 \\ &= 3abc - a^3 - b^3 - c^3 = -(a^3 + b^3 + c^3 - 3abc) \\ &= -(a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca) \\ &= -(-2)[(-2)^2 - 3(0)] = 8 \end{aligned}$$

25. For non-trivial solution, $|A|$ or $D = 0$, That is, $\begin{vmatrix} \lambda & \lambda+1 & \lambda-1 \\ \lambda+1 & \lambda & \lambda+2 \\ \lambda-1 & \lambda+2 & \lambda \end{vmatrix} = 0$

Now, $R_2 \to R_2 - R_1; R_3 \to R_3 - R_1$ gives $\begin{vmatrix} \lambda & \lambda+1 & \lambda-1 \\ 1 & -1 & 3 \\ -1 & 1 & 1 \end{vmatrix} = 0$

Also, $R_3 \to R_3 + R_2$ gives $\begin{vmatrix} \lambda & \lambda+1 & \lambda-1 \\ 1 & -1 & 3 \\ 0 & 0 & 4 \end{vmatrix} = 0$

Evaluation using third row, we get

$$4(-\lambda - \lambda - 1) = 0 \Rightarrow \lambda = -\frac{1}{2}$$

which is exactly the real value of λ.

Exercise-2 : One or More than One Answer is/are Correct

1. $f(a, b) = a(a + b)(a + 2b)$

2. $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$

$$\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ \cos^2\theta & \sin^2\theta & 1 + 2\sqrt{3}\tan\theta \end{vmatrix} = 0 \Rightarrow \tan\theta = -\frac{1}{\sqrt{3}}$$

3. $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$

$$\Delta = d^2 \begin{vmatrix} -1 & -1 & 3 \\ -1 & 2 & -1 \\ a+2d & a & a+d \end{vmatrix} = -d^2(13d + 12a)$$

4. $\begin{vmatrix} 1-\lambda & 3 & -4 \\ 1 & -(3+\lambda) & 5 \\ 3 & 1 & -\lambda \end{vmatrix} = 0$

5. $D(x) = \begin{vmatrix} x^2 + 4x - 3 & 2x + 4 & 13 \\ 2x^2 + 5x - 9 & 4x + 5 & 26 \\ 8x^2 - 6x + 1 & 16x - 6 & 104 \end{vmatrix}$

$C_3 \to C_3 - 4C_2, C_2 \to C_2 - 2C_1$

$$D(x) = \begin{vmatrix} 3x + 3 & 3 & 0 \\ 26x - 37 & 26 & 0 \\ 8x^2 - 6x + 1 & 16x - 6 & 104 \end{vmatrix}$$

7. $D=\begin{vmatrix} a & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & a \end{vmatrix}=0 \Rightarrow a^2-a-2=0 \Rightarrow (a-2)(a+1)=0$

$D_1=\begin{vmatrix} 0 & 1 & 2 \\ b & 2 & 1 \\ 0 & 1 & a \end{vmatrix} \quad D_2=\begin{vmatrix} a & 0 & 2 \\ 1 & b & 1 \\ 2 & 0 & a \end{vmatrix} \quad D_3=\begin{vmatrix} a & 1 & 0 \\ 1 & 2 & b \\ 2 & 1 & 0 \end{vmatrix}$

$a=2$ infinite solution

$a=-1, b\neq 0$ has no solution.

8. $D=\begin{vmatrix} a & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & a \end{vmatrix}=0 \quad \Rightarrow a^2-a-2=0$

$\Rightarrow (a-2)(a+1)=0$

$D=\begin{vmatrix} 0 & 1 & 2 \\ b & 2 & 1 \\ 0 & 1 & a \end{vmatrix} \quad D=\begin{vmatrix} a & 0 & 2 \\ 1 & b & 1 \\ 2 & 0 & a \end{vmatrix} \quad D=\begin{vmatrix} a & 1 & 0 \\ 1 & 2 & b \\ 2 & 1 & 0 \end{vmatrix}$

$a=2$ infinite solution

$a=-1, b\neq 0$ has no solution.

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 3

$D=\begin{vmatrix} 2 & \lambda & 6 \\ 1 & 2 & \mu \\ 1 & 1 & 3 \end{vmatrix}=(\lambda-2)(\mu-3); \qquad D_1=\begin{vmatrix} 8 & \lambda & 6 \\ 5 & 2 & \mu \\ 4 & 1 & 3 \end{vmatrix}=(\lambda-2)(4\mu-15)$

$D_2=\begin{vmatrix} 2 & 8 & 6 \\ 1 & 5 & \mu \\ 1 & 4 & 3 \end{vmatrix}=0; \qquad D_3=\begin{vmatrix} 2 & \lambda & 8 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{vmatrix}=(\lambda-2)$

Exercise-4 : Subjective Type Problems

2. $R_1 \to R_1+R_2+R_3$

$$3\begin{vmatrix} a_1+b_1+c_1 & a_2+b_2+c_2 & a_3+b_3+c_3 \\ 2b_1+c_1 & 2b_2+c_2 & 2b_3+c_3 \\ 2c_1+a_1 & 2c_2+a_2 & 2c_3+a_3 \end{vmatrix}$$

$$R_2 \to R_2 + R_1 - R_3 = 9\begin{vmatrix} a_1+b_1+c_1 & a_2+b_2+c_2 & a_3+b_3+c_3 \\ b_1 & b_2 & b_3 \\ 2c_1+a_1 & 2c_2+a_2 & 2c_3+a_3 \end{vmatrix}$$

Now, operate as $R_3 \to R_3 - R_1 + R_2$

then $R_1 \to R_1 - R_2 - R_3$

3. Let $f(x) = \begin{vmatrix} (1+x)^2 & (1+x)^4 & (1+x)^6 \\ (1+x)^3 & (1+x)^6 & (1+x)^9 \\ (1+x)^4 & (1+x)^8 & (1+x)^{12} \end{vmatrix}$

Coefficient of 'x' is $f'(0)$.

$$f'(x) = \begin{vmatrix} 2(1+x)^2 & 4(1+x)^3 & 6(1+x)^5 \\ (1+x)^3 & (1+x)^6 & (1+x)^9 \\ (1+x)^4 & (1+x)^8 & (1+x)^{12} \end{vmatrix} + \begin{vmatrix} (1+x)^2 & (1+x)^4 & (1+x)^6 \\ 3(1+x)^2 & 6(1+x)^5 & 9(1+x)^8 \\ (1+x)^4 & (1+x)^8 & (1+x)^{12} \end{vmatrix}$$

$$+\begin{vmatrix} (1+x)^2 & (1+x)^2 & (1+x)^6 \\ (1+x)^3 & (1+x)^6 & (1+x)^9 \\ 4(1+x)^3 & 8(1+x)^7 & 12(1+x)^{11} \end{vmatrix}$$

Put $x=0$, $f'(0)=0$

5. For non-zero solution, $\Delta = 0$

$$\begin{vmatrix} 2 & 3 & -1 \\ 3 & 2 & k \\ 4 & 1 & 1 \end{vmatrix} = 0 \Rightarrow k = 0$$

Now, let $x = \lambda$

So, $\quad y = -\dfrac{3\lambda}{2}, \quad z = -\dfrac{5\lambda}{2}$

$\Rightarrow$ Minimum positive integer value of z is at $\lambda = -2$; $z = 5$

6. $\begin{vmatrix} 2a & -2 & 3 \\ 1 & a & 2 \\ 2 & 0 & a \end{vmatrix} = 0 \Rightarrow a = 2$

7. Let three terms be $A-d, A, A+d$.

$\Rightarrow \quad A^4 = (A-d)^2(A+d)^2 = A^4 + d^4 - 2A^2d^2$

$\Rightarrow \quad d = \pm\sqrt{2}A, r = 3+2\sqrt{2}$ or $r = 3-2\sqrt{2}$

8. $\Delta_3 = \begin{vmatrix} 3a_1+b_1 & 3a_2+b_2 & 3a_3+b_3 \\ 3b_1 & 3b_2 & 3b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{vmatrix} = \begin{vmatrix} 3a_1 & 3a_2 & 3a_3 \\ 3b_1 & 3b_2 & 3b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{vmatrix} = 27\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$\Delta_2=\begin{vmatrix}6a_1 & 2a_2 & 2a_3\\ 3b_1 & b_2 & b_3\\ 12c_1 & 4c_2 & 4c_3\end{vmatrix}=24\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

9. $$\Delta=\begin{vmatrix}1 & \cos\theta & 1\\ -\cos\theta & 1 & \cos\theta\\ -1 & -\cos\theta & 2\end{vmatrix}=3(1+\cos^2\theta)$$

Its minimum value $=3$

10. $$D=\begin{vmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 2 & 5 & \lambda\end{vmatrix}=\lambda-8=0\Rightarrow\lambda=8$$

$$D_3=\begin{vmatrix}1 & 1 & 6\\ 1 & 2 & 14\\ 2 & 5 & \mu\end{vmatrix}=\mu-36=0\quad\Rightarrow\mu=36$$

11. $$n\sin 2\pi\left(1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}\cdots\frac{1}{\underline{|N}}\right)\underline{|n}$$

$$n\sin 2\pi\left(1+\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\ldots\frac{1}{(n+1)(n+2)\ldots(N)}\right)$$

Using $\sin(2n\pi+\theta)=\sin\theta$

$$=n(2\pi)\left(\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\ldots\frac{1}{(n+1)(n+2)\ldots N}\right)$$

Using $\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta}=1$

$$=2\pi$$

12. $$\begin{vmatrix}\cos\theta & \sin\theta & \cos\theta\\ \sin\theta & \cos\theta & \sin\theta\\ \cos\theta & \sin\theta & -\cos\theta\end{vmatrix}=0\quad\Rightarrow -2\cos\theta\cos 2\theta=0$$

* * *

11

COMPLEX NUMBERS

Exercise-1 : Single Choice Problems

2. $\arg(z-2-7i)=\cot^{-1}(2) \Rightarrow \dfrac{y-7}{x-2}=\dfrac{1}{2}$

$\arg\left(\dfrac{z-5i}{z+2-i}\right)=\pm\dfrac{\pi}{2} \qquad \Rightarrow x(x+2)+(y-5)(y-1)=0$

4. $z_1^2+z_2^2=z_1z_2$

5. Let $\omega=re^{i\theta}$ then $z=\dfrac{1}{r}e^{i(\pi/2+\theta)}$

$$\bar{z}\omega=\frac{1}{r}e^{-i(\pi/2+\theta)}r\cdot e^{i\theta}=e^{-i\pi/2}$$

6. $a\displaystyle\sum_{r=1}^{n} r\omega^{r-1}+b\sum_{r=1}^{n}\omega^{r-1}=a(1+2\omega+3\omega^2+\ldots\ldots+n\omega^{n-1})+b(1+\omega+\omega^2+\ldots\ldots+\omega^{n-1})$

$$=a\left\{\frac{1+\omega+\omega^2+\ldots\ldots+\omega^{n-1}}{1-\omega}-\frac{n\omega^n}{1-\omega}\right\}+b(0)$$

$$=a\left(0-\frac{n}{1-\omega}\right)+0$$

8. $z^4+z^3+2=0$ has roots z_1, z_2, z_3 and z_4.

$\Rightarrow\ (z-1)^4+2(z-1)^3+32=0$ has roots $(2z_1+1), (2z_2+1), (2z_3+1)$ and $(2z_4+1)$

9. $\arg\left(\dfrac{z-6-3i}{z-3-6i}\right)=\dfrac{\pi}{4}$

$\Rightarrow\ (x-6)^2+(y-6)^2=9$

11. $|iz+z_1|=|i||z-iz_1|=|z-iz_1|$

Maximum distance of $iz_1(-3+5i)$ from z is $2+\sqrt{3^2+(5-1)^2}=7$

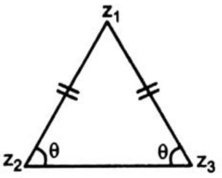
12.

$$\frac{z_1 - z_2}{z_3 - z_2} = \frac{|z_1 - z_2|}{|z_3 - z_2|} e^{i\theta}$$

$$\frac{z_1 - z_3}{z_2 - z_3} = \frac{|z_1 - z_3|}{|z_2 - z_3|} e^{-i\theta}$$

$$\arg\left(\frac{z_1 - z_2}{z_3 - z_2} + \frac{z_1 - z_3}{z_3 - z_2}\right) = \arg\left(\left|\frac{z_1 - z_2}{z_3 - z_2}\right| e^{i\theta} - \left|\frac{z_1 - z_3}{z_2 - z_3}\right| e^{-i\theta}\right)$$

$$= \pm\frac{\pi}{2}$$



13. $\frac{z_2}{z_1} = \frac{3}{2} e^{i\pi/3}$

$$\left|\frac{z_1 + z_2}{z_1 - z_2}\right| = \left|\frac{1 + \frac{3}{2} e^{i\pi/3}}{1 - \frac{3}{2} e^{i\pi/3}}\right| = \left|\frac{\left(2 + 3\cos\frac{\pi}{3}\right) + 3i\sin\frac{\pi}{3}}{2 - 3\cos\frac{\pi}{3} - 3i\sin\frac{\pi}{3}}\right|$$

$$= \sqrt{\frac{\left(\frac{7}{2}\right)^2 + \left(\frac{3\sqrt{3}}{2}\right)^2}{\left(\frac{1}{2}\right)^2 + \left(\frac{3\sqrt{3}}{2}\right)^2}} = \sqrt{\frac{49 + 27}{1 + 27}} = \frac{\sqrt{133}}{7}$$

14. $z_1 z_2 z_3 = -c$

$\Rightarrow$ $1 = |c| \Rightarrow |c| = 1$

$|z_1 + z_2 + z_3| \le |z_1| + |z_2| + |z_3|$

$|a| \le 3$

$|b| = |z_1 z_2 + z_2 z_3 + z_3 z_1| \le |z_1 z_2| + |z_2 z_3| + |z_3 z_1|$

$\Rightarrow$ $|b| \le 3$

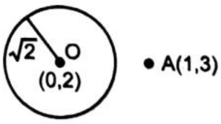
15. $\frac{1}{2} \le |z| \le 4$

$$\left|z + \frac{1}{z}\right| = \sqrt{\left(\left(r + \frac{1}{r}\right)\cos\theta\right)^2 + \left(\left(r - \frac{1}{r}\right)\sin\theta\right)^2} = \sqrt{r^2 + \frac{1}{r^2} + 2(\cos\theta - \sin\theta)}$$

16. $|3 + i(z - 1)| = |z - 1 - 3i|$

Maximum distance of A from $(z) = OA + r$

$$= \sqrt{1 + 1} + \sqrt{2} = 2\sqrt{2}$$



17. $x^2 - (\sqrt{2}i)x - 1 = 0$

$$x = \frac{\sqrt{2}i \pm \sqrt{-2 + 4}}{2} = \frac{1}{\sqrt{2}}(\pm 1 + i)$$

$$x = cis\,\frac{\pi}{4}, cis\,\frac{3\pi}{4}$$

$$x^{2187} = cis\,\frac{3\pi}{4},\ cis\,\frac{\pi}{4}$$

$$\frac{1}{x^{2187}} = cis\left(\frac{-3\pi}{4}\right),\ cis\left(-\frac{\pi}{4}\right) \Rightarrow x^{2187} - \frac{1}{x^{2187}} = 2i\sin\frac{3\pi}{4}, 2i\sin\frac{\pi}{4} = \sqrt{2}\,i$$

18. $1\cdot\frac{(1+z^9)}{1+z} = 0, z \neq -1$

$\Rightarrow \quad z^9 = -1$

$\Rightarrow \quad re^{i\theta} = e^{\frac{i(2n+1)\pi}{9}}$, $n = 1, 2, \ldots\ldots 8$

19. Let $P(re^{i\alpha})$ & $Q(re^{i\beta})$

Point of intersection of tangents at 'α', 'β' to circle $x^2 + y^2 = r^2$ is

$$\left(r\cdot\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}} - i\frac{r\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right) = \frac{re^{i\left(\frac{\alpha+\beta}{2}\right)}}{\cos\frac{\alpha-\beta}{2}} = \frac{2\omega_1\omega_2}{\omega_1+\omega_2}$$

20. $|z_1 - z_2|^2 + |z_2 - z_3|^2 + |z_3 - z_1|^2 = 2(4+9+16) - 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a})$

where $\vec{a}, \vec{b}, \vec{c}$ are position vectors of points z_1, z_2, z_3

$\Rightarrow$ Maximum value $= 58 - 2(6+12+8)\left(-\frac{1}{2}\right) = 84$

21. We have $\quad Z = \frac{7+i}{3+4i}$

Simplifying (*i.e.*, rationalizing the denominator), we get

$$\frac{7+i}{3+4i}\times\frac{3-4i}{3-4i} = \frac{21+4-28i+3i}{9+16}$$

$$= \frac{25-25i}{25} = 1-i$$

Therefore, $\quad \left(\frac{7+i}{3+4i}\right)^{14} = (1-i)^{14}$

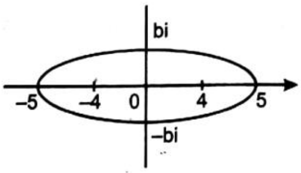
$$= [(1-i)^2]^7 = (1+i^2-2i)^7$$

$$= (+2^7)i$$

22. $|Z-4| + |Z+4| = 10$

$PS + PS' = 2a$

which implies that foci at 4 and −4 and $a = 5$ as shown in the following figure.



Now, $b^2 = 25(1-e^2) = 25-(5e)^2$

$$= 25 - 16 = 9$$

$\Rightarrow$ $$b = 3$$

Z lies on the ellipse circumference $|Z|$ denotes the distance from the origin. Therefore,

$$|Z|_{\max} = 5$$
$$|Z|_{\min} = 3$$

Thus, the difference between the maximum and the minimum values of $|Z|$ is

$$|Z|_{\max} - |Z|_{\min} = 5 - 3 = 2$$

Exercise-2 : One or More than One Answer is/are Correct

1. Let $z_1 = re^{i\theta}$ and $z_2 = re^{i\phi}$

$$|z_1 + z_2| = |z_1|$$

$\Rightarrow$ $|e^{i\theta} + e^{i\phi}| = |e^{i\theta}| = 1$

$\Rightarrow$ $(\cos\theta + \cos\phi)^2 + (\sin\theta + \sin\phi)^2 = 1$

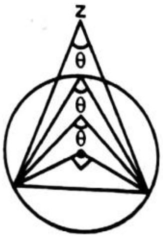
$\Rightarrow$ $\cos(\theta-\phi) = -\dfrac{1}{2}$ $\quad\Rightarrow\quad$ $\theta - \phi = \dfrac{2\pi}{3}$ or $-\dfrac{2\pi}{3}$

$$\frac{z_1}{z_2} = e^{i(\theta-\phi)} = e^{i2\pi/3} \text{ or } e^{-i2\pi/3}$$

2. (a) If $\arg\left(\dfrac{z_1}{z_2}\right) = \dfrac{\pi}{2}$ then z_1 and z_2 subtend right-angle at circumcentre origin.

$\therefore$ the chord joining z_1 and z_2 will subtend an angle θ at 'z' such that

$$\begin{cases} \theta = \pi/4 & \text{if } |z| = 1 \\ \theta < \pi/4 & \text{if } |z| > 1 \\ \theta > \pi/4 & \text{if } |z| < 1 \end{cases}$$



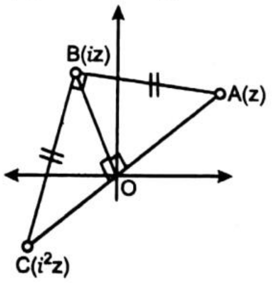
(b) $|z_1z_2 + z_2z_3 + z_3z_1| = |z_1|\cdot|z_2|\cdot|z_3|\left|\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3}\right|$

$$= |\overline{z_1 + z_2 + z_3}|$$

(c) $\left(\dfrac{(z_1+z_2)(z_2+z_3)(z_3+z_1)}{z_1z_2z_3}\right) = \left(\dfrac{\overline{(z_1+z_2)(z_2+z_3)(z_3+z_1)}}{z_1z_2z_3}\right)$

(d) The triangle formed by joining z_1, z_3 and z_2 is isosceles and right angled at z_3.

3.



Method I : Multiplying a complex number by i rotates a vector for z in the anticlockwise direction by an angle of 90°.

$$\therefore \qquad \angle AOB = \angle BOC = 90^\circ$$

As shown in figure, the ΔABC is a right angled isosceles triangle.

Method II : Let z, iz, i^2z are vertices A, B and C of the triangle ABC.

$$\therefore \qquad |AB| = |BC| \text{ also } |AB|^2 + |BC|^2 = |AC|^2$$

Since, $\qquad |AB| = |BC|$ also $|AB|^2 + |BC|^2 = |AC|^2$

$\therefore$ the ΔABC is a right angled isosceles triangle.

7. $(z+i)^4 = 1+i$

$$z = -i + 2^{1/8}\cos\left(\frac{\pi}{8} + \frac{2m\pi}{4}\right)$$

$$\text{Square side length} = \left(\frac{2^{1/8}\cdot 2}{\sqrt{2}}\right)$$

8. $z = 4\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

$$\text{Roots} = 4^{1/4}\cos\left(\frac{2m\pi}{4} - \frac{60^\circ}{4}\right)$$

$m = 0, 1, 2, 3 \qquad m = 1$

9. $a + b\omega + c\omega^2 = \alpha$

$a + b\omega^2 + c\omega = \overline{\alpha}$

$$|\alpha| = 1 \Rightarrow \left|a + b\left(-\frac{1}{2} + \frac{\sqrt{3}\,i}{2}\right) + c\left(-\frac{1}{2} - \frac{\sqrt{3}\,i}{2}\right)\right| = 1$$

10. Check option for $z = \omega$

$\omega^{62} + \omega + 1 = 0 \qquad \omega^2 + \omega + 1 = 0$

$\omega^{155} + \omega + 1 = 0 \qquad \omega^2 + \omega + 1 = 0$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 2

Sol. $f(z)+\overline{f(z)}=f(\bar{z})+\overline{f(\bar{z})}$

$$(\alpha z+\beta)+(\bar{\alpha}\,\bar{z}+\bar{\beta})=\alpha\,\bar{z}+\beta+\bar{\alpha}\,z+\bar{\beta}$$

$$\Rightarrow\quad (\alpha-\bar{\alpha})(z-\bar{z})=0$$

$$\Rightarrow\quad \operatorname{Im}(\alpha)=0 \qquad (\operatorname{Im}(z)\neq 0)$$

$$f(z)+\overline{f(z)}=0$$

$$\Rightarrow\quad \alpha(z+\bar{z})+(\beta+\bar{\beta})=0 \qquad (\because\ \alpha=\bar{\alpha})$$

$$\Rightarrow\quad \operatorname{Re}(\beta)=0 \qquad (\operatorname{Re}(z)=0)$$

$$|f(z)|^2>(z+1)^2$$

$$\Rightarrow\quad \alpha^2z^2+\beta^2>z^2+2z+1$$

$$\Rightarrow\quad (\alpha^2-1)z^2-2z+(\beta^2-1)>0\ \forall\ z\in R$$

Paragraph for Question Nos. 3 to 5

Sol. $|\alpha-\beta|=2\sqrt{7}$

$$\Rightarrow\quad |(\alpha+\beta)^2-4\alpha\beta|=28$$

$$\Rightarrow\quad |z_1^2-4(z_2+m)|=28$$

$$\Rightarrow\quad |m-(4+5i)|=7$$

$$\text{greatest}\,(|m|)=\sqrt{16+25}+7$$

$$\text{least}\,|m|=7-\sqrt{16+25}$$

Paragraph for Question Nos. 6 to 7

Sol. $C_1:|z-z_1|^2+|z-z_2|^2=10 \quad\Rightarrow C_1:(x-5)^2+y^2=1$

$C_2:|z-z_1|^2+|z-z_2|^2=16 \quad\Rightarrow C_2:(x-5)^2+y^2=4$

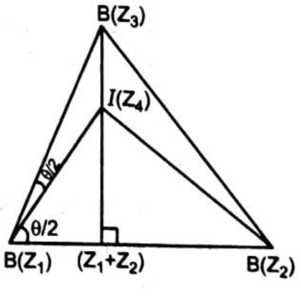
Paragraph for Question Nos. 8 to 9

Sol. $\dfrac{Z_2-Z_1}{|Z_2-Z_1|}=\dfrac{Z_4-Z_1}{|Z_4-Z_1|}\cdot e^{i\theta/2},\quad \dfrac{Z_3-Z_1}{|Z_3-Z_1|}=\dfrac{Z_4-Z_1}{|Z_4-Z_1|}e^{i\theta/2}$

$$\Rightarrow\quad \frac{(Z_2-Z_1)(Z_3-Z_1)}{|Z_2-Z_1||Z_3-Z_1|}=\frac{(Z_4-Z_1)^2}{|Z_4-Z_1|^2}e^{\theta i}$$

$$\Rightarrow\quad \frac{(Z_2-Z_1)(Z_3-Z_1)}{(Z_4-Z_1)^2}=\frac{AB\cdot AC}{(IA)^2}$$

Similarly, others.



Exercise-4 : Matching Type Problems

1. Let $BC = n$, $CA = n+1$, $AB = n+2$

(A) $\left|\arg\left(\dfrac{z_1 - z_3}{z_2 - z_3}\right)\right| = \left|2\arg\left(\dfrac{z_3 - z_1}{z_2 - z_1}\right)\right| = \angle C = 2\angle A$

$\therefore \quad \dfrac{\sin C}{c} = \dfrac{\sin A}{a} \Rightarrow \dfrac{\sin 2A}{n+2} = \dfrac{\sin A}{n}$

$\Rightarrow \quad \cos A = \dfrac{n+2}{2n} \Rightarrow \dfrac{(n+2)^2 + (n+1)^2 - n^2}{2(n+2)(n+1)} = \dfrac{n+2}{2n}$

$\Rightarrow \quad n(n^2 + n + 5) = (n^2 + 3n + 2)(n+2) \Rightarrow n^2 - 3n - 4 = 0 \Rightarrow n = 4$

$\therefore$ biggest side $= n + 2 = 6$

(B) $(\vec{c} - \vec{a})\cdot(\vec{b} - \vec{c}) = 0 \Rightarrow \angle C = 90^\circ \Rightarrow a^2 + b^2 = c^2$

$\Rightarrow \quad n^2 + (n+1)^2 = (n+2)^2 \Rightarrow n = 3$

$\therefore$ Area $= \dfrac{1}{2}\cdot 3 \cdot 4 = 6 = \Delta$

$\therefore \quad |\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}| = 2\Delta = 12$

(C) $\dfrac{|a_1 b_2 - a_2 b_1|}{|a_1 a_2 + b_1 b_2|} = \tan A = \dfrac{4}{3} \qquad \therefore \cos A = \dfrac{3}{5}$

$\therefore \quad \dfrac{(n+2)^2 + (n+1)^2 - n^2}{2(n+2)(n+1)} = \dfrac{3}{5}$

$\Rightarrow \quad 5(n^2 + 6n + 5) = 6(n^2 + 3n + 2)$

$\Rightarrow \quad n^2 - 12n - 13 = 0 \Rightarrow n = 13$

$\therefore \quad S - c = \dfrac{1}{2}(a + b - c) = \dfrac{1}{2}(13 + 14 - 15) = 6$

(D) Altitudes are in H.P. ⇔ sides are in A.P.

Also, $b > a + c, a > b + c, c > a + b \Rightarrow$ least value of $a = 2$

$\therefore$ least value of $b = 3$

3. (A) $\{0, 1, \omega + 1\}^m = \{0, 1, -\omega^2\}^m$

$0, 1, -1, -\omega^2, -\omega, \omega$

(B) $2\omega, (x^2 - x + 10) = 0$ roots are $2 + 3\omega,\ 2 + 3\omega^2$

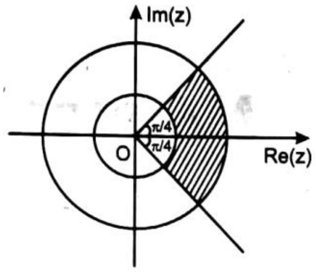
Last number is 3.

(C) Central angle = 60° Equilateral Δ

(D) Put $z = 1$ $\quad z_1 = 1,\ z_2 = \omega,\ z_3 = \omega^2$

Exercise-5 : Subjective Type Problems

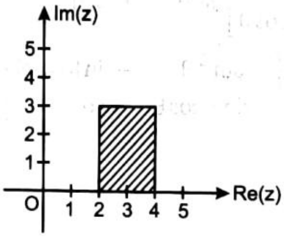
1.


Im(z)
Re(z)
O
π/4
π/4

$2 \le |z| \le 4$

Probability $= \dfrac{1}{4}$

2.


Im(z)
5
4
3
2
1
O
1
2
3
4
5
Re(z)

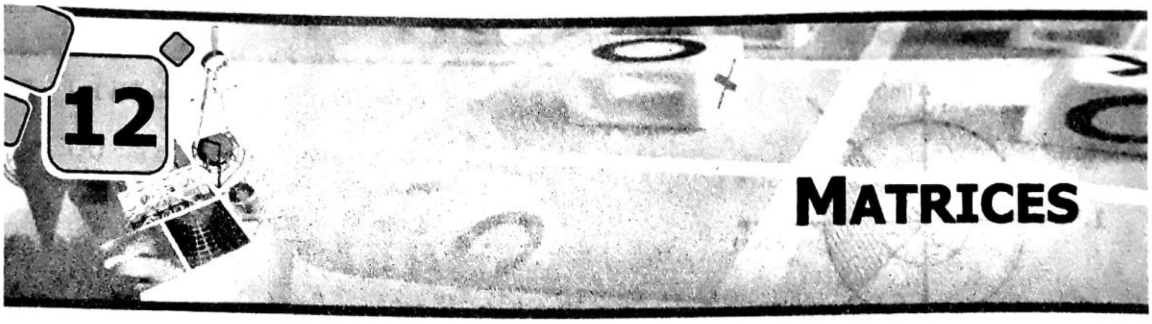
3. $z + \bar{z} = 2|z-1| \quad \Rightarrow y^2 = 2x - 1$

$\arg(z_1 - z_2) = \dfrac{\pi}{4} \Rightarrow y_1 - y_2 = x_1 - x_2$

$y_1^2 - y_2^2 = 2(x_1 - x_2) = 2(y_1 - y_2) \quad \Rightarrow y_1 + y_2 = 2 \quad (y_1 \ne y_2)$

□□□

Chapter 12 – Matrices



12

MATRICES

Exercise-1 : Single Choice Problems

1. $A = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} [\cos\theta \quad \sin\theta] + \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix} [\sin\theta \quad -\cos\theta]$

$$= \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} + \begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{bmatrix} = I$$

2. $|A| = \begin{vmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{vmatrix} = 1$

$$A^2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3. $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$\det(\text{adj}(\text{adj}(A))) = |A|^4 = 27^4$

$$\left\{ \frac{27^4}{5} \right\} = \frac{1}{5}$$

4. $A^{-1}B^{-1} = B^{-1}A^{-1} \Rightarrow C = (A^{-1} + B^{-1})^5 = (I)^5$

5. $A^4 = I \Rightarrow A(A^3) = I$

7. $(\text{adj } A)A = |A|I$

$$|A| = xyz - 8x - 4y - 3z + 28 = 2\lambda - \lambda = \lambda$$

8. $(x-2) + (x^2 - x + 3) + (x - 7) = 0$

$x^2 + x - 6 = 0 \quad \Rightarrow \quad (x+3)(x-2) = 0$

9. $A=\begin{bmatrix}-1 & 3\\ 3 & 0\end{bmatrix} \Rightarrow A^{-1}=\frac{1}{9}\begin{bmatrix}0 & 3\\ 3 & 1\end{bmatrix}$

10. $\begin{bmatrix}1 & -\tan\theta\\ \tan\theta & 1\end{bmatrix}\begin{bmatrix}\cos^2\theta & -\sin\theta\cos\theta\\ \sin\theta\cos\theta & \cos^2\theta\end{bmatrix}=\begin{bmatrix}a & -b\\ b & a\end{bmatrix}$

$\Rightarrow \quad a=\cos 2\theta,\ b=\sin 2\theta$

11. $P^2=I-P$

or $P^3=P-P^2=2P-I$

or $P^4=2I-3P$

or $P^5=-3I+5P$

or $P^6=5I-8P$

12. $|\text{adj}(\text{adj}(A))|=|A|^{(n-1)^2}$

$\Rightarrow \quad |A|=x+y+z=12$

$x\ge 1, y\ge 1, z\ge 1$

$\Rightarrow \quad {}^{11}C_2=55$

13. Let $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$; $\text{adj}(A)=\begin{bmatrix}d & -c\\ -b & a\end{bmatrix}^T=\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$; $\text{adj}(\text{adj}(A))=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$

14. $M=A^{2m}\cdot A^{-1}$

$M=\dfrac{A^{2m+1}}{a^2+b^2}$

If $A^2=(a^2+b^2)\cdot I \Rightarrow A^{2m}=(a^2+b^2)^m\cdot I$

$A^{2m+1}=(a^2+b^2)^m\cdot A$

15. $A^2+5A+6I=I$

$(A+2I)(A+3I)=I$

$\Rightarrow$ $A+2I$ and $A+3I$ are inverse of each other.

16. $AB=\begin{bmatrix}3 & -5\\ 7 & -12\end{bmatrix}\begin{bmatrix}12 & -5\\ 7 & -3\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$

17. $\text{adj}(A)=\begin{bmatrix}3 & -2\\ -2 & 2\end{bmatrix}$

18. $AA^1=I$

$$\begin{bmatrix}\cos\theta & 2\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ 2\sin\theta & \cos\theta\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}\cos^2\theta+4\sin^2\theta & 3\sin\theta\cos\theta\\ 3\sin\theta\cos\theta & \sin^2\theta+\cos^2\theta\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\Rightarrow \sin\theta=0$$

20. If $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $P = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, $Q = P^T AP$, we have

$$PQ^{2014}P^T = \underbrace{P(P^T AP)(P^T AP)\ldots(P^T AP)P^T}_{2014 \text{ times}}$$

$$= (PP^T)A(PP^T)A(PP^T)\ldots(PP^T)A(PP^T)$$

Matrix multiplication is associative.

$$PP^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

Hence, $PQ^{2014}P^T = A^{2014}$

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \Rightarrow A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 8 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad A^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix} \text{ and } A^{2014} = \begin{pmatrix} 1 & 4028 \\ 0 & 1 \end{pmatrix}$$

21. $\left|adj\left(\frac{M}{2}\right)\right| = \left|\frac{M}{2}\right|^2 = \left(\frac{1}{8}|M|\right)^2$

22. $|A^{-1}| = \frac{1}{|A|} = \frac{1}{5}$

$|(AB)^T| = |AB| = |A \cdot (adj\ A)| = |A| \cdot |adj\ (A)| = 5 \times 5^2 = 5^3$

$\therefore ||A^{-1}|(AB)^T| = |\frac{1}{5}(AB)^T| = \frac{1}{5^3}|AB| = 1$

Exercise-2 : One or More than One Answer is/are Correct

3. $A_\alpha A_\beta = A_{\alpha+\beta}$

Also, $A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

and $A_\alpha A_{-\alpha} = A_{\alpha-\alpha} = A_0 = I$

we get $A_\alpha^{-1} = A_{-\alpha}$

However, $A_\alpha^{-1} = -A_\alpha$ and $A_\alpha^2 = -I$ do not hold.

4. $A(A^2 - I) - 2(A^2 - I) = 0$

$$(A^2 - I)(A - 2I) = 0$$

Exercise-3 : Matching Type Problems

1. (A) Possible non-negative value of $|A| = 2, 4, 8$

(B) Sum is 0.

(C) $|\text{adj}(\text{adj}(\text{adj } A)))| = |A|$

least absolute value of $|A| = 2$

$\Rightarrow \quad |A| = \pm 2$

(D) least $|A| = -8$

$$|4A^{-1}| = \frac{16}{|A|} = -2$$

2. (A) Since A is idempotent, $A^2 = A^3 = A^4 = \ldots\ldots = A$. Now,

$$\begin{aligned}(A + I)^n &= I + {}^nC_1A + {}^nC_2A^2 + \ldots\ldots + {}^nC_nA^n \\ &= I + {}^nC_1A + {}^nC_2A + \ldots\ldots + {}^nC_nA \\ &= I + ({}^nC_1 + {}^nC_2 + \ldots\ldots + {}^nC_n)A \\ &= I + (2^n - 1)A\end{aligned}$$

$\Rightarrow \quad 2^n - 1 = 127 \Rightarrow n = 7$

(B) We have,

$$\begin{aligned}&(I - A)(I + A + A^2 + \ldots\ldots + A^7) \\ &= I + A + A^2 + \ldots\ldots + A^7 + (-A - A^2 - A^3 - A^4 \ldots\ldots - A^8) \\ &= I - A^8 \\ &= I \qquad \text{(if } A^8 = 0)\end{aligned}$$

(C) Here matrix A is skew-symmetric and since $|A| = |A^T| = (-1)^n|A|$, so $|A|(1 - (-1)^n) = 0$. As n is odd, hence $|A| = 0$. Hence A is singular.

(D) If A is symmetric, A^{-1} is also symmetric for matrix of any order.

5. (A) $\displaystyle \frac{1}{n}\sum_{r=1}^{n}\left(\frac{1}{\sqrt{\frac{r}{n}}}\right) = \int_0^1 \frac{1}{\sqrt{x}}\,dx$

(B) $D = 4\cos t \cos 2t$

(C) $3x^2 + 2px + g < 0$

$$f\left(-\frac{5}{3}\right) = 0 \qquad f(-1) = 0$$

(D) $(2^x - 2)^2 + 1 + ||b-1|-3| = |\sin y|$

$b - 1 = \pm 3$

$|\sin y| = 1$

Exercise-4 : Subjective Type Problems

1. $(AB)^2 = AB \cdot AB = A^3B^2$

$(AB)^3 = (AB)^2 \cdot AB = A^3B^2 \cdot AB = A^7B^3$

$(AB)^4 = (AB)^3 \cdot AB = A^7B^3 \cdot AB = A^{15}B^4 \quad \Rightarrow (AB)^{10} = A^{1023}B^{10}$

2. $$l = \lim_{n\to\infty} 18\left(\frac{3}{3^2} + \frac{3^2}{3^4} + \frac{3^3}{3^6} + \ldots\ldots\right) = 18 \times \frac{1}{3\left(1-\frac{1}{3}\right)} = 9$$

$$m = \lim_{n\to\infty} 12\left(\frac{2}{2^2} + \frac{2^2}{2^4} + \ldots\ldots\right) = 12 \times \frac{1}{2\left(1-\frac{1}{2}\right)} = 12$$

4. $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1b_2c_3 - a_1c_2b_3 \quad \text{...six elements}$$

All cannot be simultaneously 1.

5. First element of matrix $A_{10} = 286$ (10^{th} of sequence 1, 2, 6, 15, ...)

Trace of $A_{10} = 286 + 297 + 308 + 319 + \ldots + 385 = 3055$

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Chapter 13 – Permutation and Combinations

13

PERMUTATION AND COMBINATIONS

Exercise-1 : Single Choice Problems

1. $\underline{7}\ \underset{\overset{\uparrow}{9}}{\underline{\ \ }}\ \underset{\overset{\uparrow}{9}}{\underline{\ \ }} = 81;\quad \underset{\overset{\uparrow}{8}}{\underline{\ \ }}\ \underline{7}\ \underset{\overset{\uparrow}{9}}{\underline{\ \ }} = 72;\quad \underset{\overset{\uparrow}{8}}{\underline{\ \ }}\ \underset{\overset{\uparrow}{9}}{\underline{\ \ }}\ \underline{7} = 72$

2. $\left(\frac{8!}{3!3!2!2!}\right)\times {}^2C_1 \times 3! + \left(\frac{8!}{3!2!2!2!}\right)\times 3! = 8400$

3. Number of ways $= 6\times\left(\frac{3!}{2!}\times 3!\right) = 108$

4. ${}^4C_1 \times \frac{5!}{2!} = 240$

5. ${}^6C_2 \times 1 \times 4! = 360$

6. $x^2 - 5x + 3 = 0 \begin{cases} \alpha \\ \beta \end{cases}$

$\Rightarrow \quad \alpha + \beta = 5, \quad \alpha\beta = 3$

Sum of roots $= \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{19}{3}$

7. ${}^5C_4 \times {}^8C_6 + {}^5C_5 \times {}^8C_5 = 196$

8. $(1 + 2 + 3 + \ldots\ldots + 22)\ {}^{21}C_{10}$

9. $x = \frac{2009 \times 2008 \times 2007 + 1}{2008 \times 2007 \times 2007} = 2008 + \frac{1}{2009 \times 2007}$

$\Rightarrow \qquad [x] = 2008$

10. $N = p_1^n p_2 p_3 \ldots\ldots p_{m+1}$

No. of factors $= (n + 1)\, 2^m$

11. Number of ways $= (11)! \times 2^{12}$

12. $\underset{5}{\uparrow}\ \underset{5}{\uparrow}\ -\ -$

$m = 5 \times 5 \times {}^8C_2 \times 2!$

$\underset{4}{\uparrow}\ \underset{5}{\uparrow}\ -\ -$

$n = 4 \times 5 \times {}^8C_2 \times 2!$

13. Three different digits (not including zero)

$${}^9C_3 \times 2!$$

Two digits (not including zero)

$${}^9C_2 \times 2$$

Three digits (including zero)

$${}^9C_2 \times 1$$

14. Let no. of elements in $A = n$

No. of elements in $B = m$

$$2^n - 2^m = 1920 = 2^7 \times 15$$

$$\Rightarrow \quad n = 11,\ m = 7$$

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 15$$

15. $C\ldots\ldots\ldots = 4! = 24$

$D\ldots\ldots\ldots = 4! = 24$

$M\ldots\ldots\ldots = 4! = 24$

Ⓢ$C\ldots\ldots = 3! = 6$

Ⓢ$D\ldots\ldots = 3! = 6$

ⓈⓂⒸ$D\,W = 1$

ⓈⓂⒸ$W\,D = 1$

16. $P = $ All A's together $= \frac{5!}{3!}$; $\qquad Q = $ All B's together $= \frac{6!}{4!}$

$$n(P \cap Q) = 3!; \qquad n(P \cup Q) = \frac{5!}{3!} + \frac{6!}{4!} - 3! = 50 - 6 = 44$$

17. $5^6 \times 6^7 \times 7^8 \times 8^9 \times 9^{10} \times 10^{11} \times \ldots\ldots \times 30^{31}$

No. of zero's = no. of 5's

$$= 6 + 11 + 16 + 21 + (2 \times 26) + 31 = 137$$

18. $(x - y)(x + y) = 10 \times 337$

$$\Rightarrow \quad x - y = 10 \text{ and } x + y = 337$$

$$x = \frac{347}{2} \qquad \text{(not possible)}$$

19. Total number of different things $= n + 2$

20. Let the numbers are $10 - d, 10, 10 + d$.

$$d \in \{-9, -8, -7, \ldots\ldots, 7, 8, 9\}$$

22. $m = 2 \times 5! \times 5!$

$n = 4! \times 5!$

23. Total ways $= 4 \times 4! = 96$

25. ${}^4C_2 \times 5^2 \times (21)^2 = 66150$

26. Total all letters are different.

$\Rightarrow \quad 10^5 - {}^{10}C_5 \times 5! = 69760$

29. $M = 1440$

$M = 2^5 \cdot 3^2 \cdot 5$

No. of divisions $= 6 \times 3 \times 2 = 36$

$P =$ Product of divisors $= (1440)^{18}$

$P = 2^{90} \cdot 3^{36} \cdot 5^{18}$

Hence, $x = 30$

30. **Case-1 :** All digits same $= 9$

Case-2 : Excluding zero :

(i) No's having 3 digits same : ${}^9C_2 \times {}^2C_1 \times \frac{4!}{3!} = 288$

(ii) No's having 2 digits same, 2 other same : ${}^9C_2 \times \frac{4!}{2!2!} = 216$

Case-3 : Including zero :

(i) No's having 3 zero's : 9

(ii) No's having 2 zero's : ${}^9C_1 \times \frac{3!}{2!} = 27$

(iii) No's having 1 zero $= {}^9C_1 \times \frac{3!}{2!} = 27$

Hence, total no's $= 576$

31. **Case-I :** When two T's contain exactly one vowel between them,

$$5! \times ({}^5C_1 \times {}^5C_4 \times 4!) = 15 \times 5! \times 5!$$

Case-II : When two T's also contain consonant between them,

$$4! \times ({}^5C_2) \times ({}^7C_5 \times 5!) = 42 \times 5! \times 5!$$

32. $666660 \rightarrow 6$

$666633 \rightarrow \frac{6!}{4!2!}$

$666642 \rightarrow \frac{6!}{4!}$

$666444 \rightarrow \dfrac{6!}{3!3!}$

33. Five 4 runs + one 0 run $= \dfrac{6!}{5!}$

Four 4 runs + two 2 runs $= \dfrac{6!}{4!2!}$

Three 4 runs + two 3 runs + one 2 runs $= \dfrac{6!}{3!2!}$

Two 4 runs + four 3 runs $= \dfrac{6!}{2!4!}$

$\Rightarrow \quad N = 96$

34. ${}^7C_2 = 21$

35. $x_1 + x_2 + x_3 + x_4 + x_5 = 101$

Let $x_1 = 2k_1 + 1, x_2 = 2k_2 + 1, x_3 = 2k_3 + 1, x_4 = 2k_4 + 1, x_5 = 2k_5 + 1$

$\Rightarrow \quad k_1 + k_2 + k_3 + k_4 + k_5 = 48; \quad {}^{48+5-1}C_{5-1}$

36. Total ways = (largest number is 4)

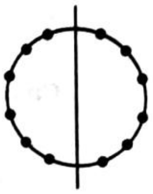
$6^4 - (4^4 - 3^4) = 1121$

37. ${}^6C_3 \times 4!$

38. If two points are selected from one side of main diagonal $= {}^6C_2$.

Then other two points are selected on other side of main diagonal = 1.

Total ways $= {}^6C_2 \times 1 = 15$



39. $(9 - x_1) + (9 - x_2) + (9 - x_3) + (9 - x_4) + (9 - x_5) = 43$

$\Rightarrow \quad x_1 + x_2 + x_3 + x_4 + x_5 = 2$

Number of ways $= {}^{2+5-1}C_{5-1} = {}^6C_4 = 15$

Exercise-2 : One or More than One Answer Is/are Correct

1. **Case-I :** All five letters are different.

$= 5!$

Case-II : Two letters are same and remaining are different.

$${}^3C_1 \times {}^4C_3 \times \frac{5!}{2!} = 720$$

Case-III : Two alike, two other alike and remaining different.

$${}^3C_2 \times {}^3C_1 \times \frac{5!}{2!2!} = 270$$

Total number of words = 1110

2. $\sum\limits_{k=0}^{100} {}^{100}C_k(x-2)^{100-k}\cdot 3^k=(x+1)^{100}$

$$\text{Coeff. of } x^{50}={}^{100}C_{50}$$

3. $\dfrac{\text{Total}-\text{Row 1}-\text{Row 2}}{\lfloor\underline{2}}$ $\{\lfloor\underline{2}\ \text{for } N\}$

$\dfrac{{}^8C_5\lfloor\underline{6}-\lfloor\underline{6}-\lfloor\underline{6}}{\lfloor\underline{2}}$

4. = (four odd) + (4 even) + (3 even + 1 odd) + (2 even + 2 odd)

$={}^5C_4\times 4!+{}^4C_4\times 4!+{}^4C_3\times {}^5C_1\times 4!+{}^4C_2\times {}^5C_2\times 4\times 4$

$=1584$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 2

1. 0

2. Digit 6 always come at last three place digit 5 always come at last four place and digit 4 always come at last five place.

$${}^3C_1\times {}^3C_1\times {}^3C_1\times 3!=162$$

Exercise-4 : Matching Type Problems

1. (A) $\dfrac{6!}{2!}\times {}^7C_2=7560$ (B) $5!\times {}^6C_2=1800$

(C) $7560-1800=5760$ (D) $4!\times {}^5C_4\cdot\dfrac{4!}{2!2!}=720$

2. (A) Total ways – (No repeating letter is at odd position)

$$\frac{11!}{2!2!2!}-0=\frac{11!}{(2!)^3}$$

(B) $\dfrac{7!}{2!2!}\times {}^8C_4\times\dfrac{4!}{2!}=210\times 7!$

(C) $\enclose{circle}{MM}\ \enclose{circle}{TT}\ \ HEICS$

$7!\times {}^8C_2\times 1=28\times 7!$

(D) $\left(\dfrac{4!}{2!}\right)\times\left(\dfrac{7!}{2!2!}\right)=\dfrac{4!\,7!}{(2!)^3}$

Exercise-5 : Subjective Type Problems

1. ${}^9C_4 \times {}^5C_4 = 630$

2. ${}^9C_2 \times \frac{7!}{2!2!} = \frac{9!}{8}$

4. ${}^{10}C_3 - {}^8C_3 = 64$

5. **Case-I:** If Ravi is include.

$${}^7C_5 \times {}^9C_8 = 189$$

Case-II: If Ravi is not include.

$${}^7C_6 \times [{}^8C_7 + {}^9C_8] = 119$$

Total number of ways $= 308$

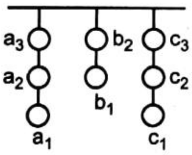
6. ${}^6C_4 - {}^4C_2 = 9$

7. $5! - (1 + {}^5C_2 \times 1) = 109$

8. Let other two sides are a and b.

$\therefore \quad a + b > 11 \qquad 0 < a \le 11, \ 0 < b \le 11$

9.



$(a_1, a_2, a_3), (b_1, b_2)$ and (c_1, c_2, c_3) are alike things so these can be arranged is

$$\frac{8!}{2!3!3!} = \frac{4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{2 \times 6} = 560$$

10. ${}^nC_2 - n = 14 \Rightarrow n = 7$

11. $x_1 + x_2 + x_3 + \ldots\ldots + x_7 + x_8 = 93$

$x_1 \ge 0, \ x_2 \ge 6, \ x_3 \ge 6, \ldots\ldots x_7 \ge 6, \ x_8 \ge 0$

$x_1 + x_2' + x_3' + \ldots\ldots + x_7' + x_8 = 57$

No. of ways $= {}^{64}C_7$

12. ${}^4C_4({}^2C_1)^4 = 16$

13. Let x_1 objects of one type

x_2 objects of second type

x_3 objects of third type

$x_1 + x_2 + x_3 = 3n$

$0 \le x_1 \le 2n, 0 \le x_2 \le 2n, 0 \le x_3 \le 2n$

Number of ways = ${}^{3n+2}C_2 - 3 \times {}^{n+1}C_2 = 3n^2 + 3n + 1$

14. $x + y + z + w = 15$

$x \ge 0, y \ge 6, z \ge 2, w \ge 1$

$x + y' + z' + w' = 6$

Number of ways = ${}^9C_3 = 84$

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Chapter 14 – Binomial Theorem

14

BINOMIAL THEOREM

Exercise-1 : Single Choice Problems

1. Let $x = 2^{\frac{153}{2}}$

$$\alpha = x^2 + \sqrt{2}\,x + 1$$

Let $y = 2^{204}$

$$\beta = y^2 - y + 1$$

$$N = x^{16} - 1$$

$$N = (x^4 - 1)(x^4 + 1)(x^8 + 1)$$

$$N = (x^4 - 1)(x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)(x^8 + 1)$$

$$N = y^6 - 1 = (y^3 - 1)(y^3 + 1)$$

$$= (y^3 - 1)(y + 1)(y^2 - y + 1)$$

3. $^4C_2\alpha^2 = -\,^6C_3\alpha^3$

$\Rightarrow \quad \alpha = -\dfrac{3}{10}$

5. $\alpha_n = (2 + \sqrt{3})^n$

Let $\alpha'_n = (2 - \sqrt{3})^n \Rightarrow \alpha_n + \alpha'_n = \text{integer}$

$\Rightarrow \quad [\alpha_n] + \{\alpha_n\} + \alpha'_n = \text{integer} \Rightarrow \{\alpha_n\} = 1 - \alpha'_n$

So, $\quad \lim\limits_{n\to\infty}(\alpha_n - [\alpha_n]) = \lim\limits_{n\to\infty}[1 - (2 - \sqrt{3})^n] = 1 - 0 = 0$ $\qquad (\because 0 > \{\alpha_n\}, \alpha'_n < 1)$

6. $N = {}^{20}C_7 - {}^{20}C_8 + {}^{20}C_9 - {}^{20}C_{10} + \ldots\ldots - {}^{20}C_{20}$

$$= ({}^{20}C_7 + {}^{20}C_9 + {}^{20}C_{11} + \ldots\ldots + {}^{20}C_{19}) - ({}^{20}C_8 + {}^{20}C_{10} + \ldots\ldots + {}^{20}C_{20})$$

$$= ({}^{20}C_0 + {}^{20}C_2 + {}^{20}C_4 + {}^{20}C_6) - ({}^{20}C_1 + {}^{20}C_3 + {}^{20}C_5)$$

$$= (1 + 190 + 4845 + 38760) - (20 + 1140 + 15504)$$

$$= 43796 - 16664 = 27132 = 3 \times 4 \times 7 \times 19 \times 17$$

7. $\log_2\left[1 + \dfrac{1}{2}(2^{12} - 2)\right] = \log_2 2^{11} = 11 \qquad [\because \Sigma\,^nC_r = 2^n]$

8. $T_{r+1} = {}^nC_r \cdot x^{n-r} \cdot y^r = {}^{12}C_r \cdot x^{12-r} \cdot \left(\dfrac{1}{x^3}\right)^r$

$$12 - 4r = 0 \Rightarrow r = 3$$

$$T_4 = {}^{12}C_3 = \frac{12 \times 11 \times 10}{3 \cdot 2 \cdot 1} = 220$$

9. $$\frac{3}{4!} + \frac{4}{5!} + \frac{5}{6!} + \ldots\ldots = \frac{1}{3!} - \frac{1}{(k+3)!}$$

$$\sum_{r=3}^{52} \frac{r}{(r+1)!} = \sum_{r=3}^{52} \left[\frac{1}{r!} - \frac{1}{(r+1)!}\right] = \frac{1}{3!} - \frac{1}{53!}$$

$\Rightarrow$ $$k = 50$$

10. $$f(x) = \sum_{r=1}^{n} [(r+1)^2\ {}^nC_r - r^2\ {}^nC_{r-1}]$$

$$f(n) = (n+1)^2 - 1$$

$$f(30) = 960$$

12. ${}^nC_1 \cdot \alpha + {}^nC_2 \cdot \alpha^2 + {}^nC_3 \cdot \alpha^3 + \ldots\ldots\ {}^nC_n \cdot \alpha^n = (1+\alpha)^n - 1$

$\left(\text{where } \alpha = e^{\frac{2\pi i}{n}} = \frac{\alpha_2}{\alpha_1}\right)$

13. $2^{30} \cdot 3^{20} = 2^{10} \cdot (6)^{20} = 1024(7-1)^{20} = 1024(7K+1) = 7k' + 1024 = 7k' + 1022 + 2$

14. ${}^{26}C_0 + {}^{26}C_1 + {}^{26}C_2 + \ldots\ldots + {}^{26}C_{26} = 2^{26}$

$\Rightarrow\ 2({}^{26}C_0 + {}^{26}C_1 + \ldots\ldots + {}^{26}C_{13}) = 2^{26} + {}^{26}C_{13}$

15. $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_{2n}x^{2n}$

differentiate w.r.t. x

$n(1 + x + x^2)^{n-1}(1 + 2x) = a_1 + 2a_2x + 3a_3x^2 + \ldots + 2n \cdot a_{2n}x^{2n-1}$

Put $x = 1$ $\quad n \cdot 3^n = a_1 + 2a_2 + 3a_3 + \ldots + 2n\,a_{2n}$ ...(1)

Put $x = \omega$ $\quad 0 = a_1 + 2a_2\omega + 3a_3\omega^2 + \ldots + 2n\,a_{2n}\omega^{2n-1}$ ...(2)

Put $x = \omega^2$ $\quad 0 = a_1 + 2a_2\omega^2 + 3a_3\omega^4 + \ldots + 2n\,a_{2n}\omega^{4n-2}$ ...(3)

(1) + (2) + (3)

$$n \cdot 3^{n-1} = a_1 + 4a_4 + 7a_7 + 10a_{10} + \ldots$$

16. ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

${}^3C_0 + {}^3C_1 + {}^4C_2 + {}^5C_3 + \ldots + {}^{99}C_{97} = {}^{100}C_{97}$

17. Last digit of $9! = 0$

Last digit of $3^{9966} = 9$

Hence last digit 9.

18. $x = T_7 = {}^nC_6(3^{1/3})^{n-6} \cdot (4^{-1/3})^6$

$y = T_{n-5} = {}^nC_{n-6}(3^{1/3})^6 \cdot (4^{-1/3})^{n-6}$

$y = 12x$

${}^nC_{n-6}(3^{1/3})^6(4^{-1/3})^{n-6} = 12 \cdot {}^nC_6(3^{1/3})^{n-6}(4^{-1/3})^6$

$\Rightarrow \quad 12 = (12^{1/3})^{12-n} \Rightarrow n = 9$

20. $t_{r+1} = {}^{15}C_r(x^2)^{15-r} \cdot \left(\frac{2}{x}\right)^r$

Coeff. of $x^{15} = {}^{15}C_5 \cdot 2^5$

Coeff. of $x^0 = {}^{15}C_{10} \cdot 2^{10}$

21. $(1+x)^2(1+y)^3(1+z)^4(1+w)^5$

General term $= {}^2C_a \; {}^3C_b \; {}^4C_d \; {}^5C_e x^{a+b+d+e}$

$$\sum_{a+b+d+e=12} {}^2C_a \times {}^3C_b \times {}^4C_d \times {}^5C_e = {}^{14}C_{12} \text{ or } {}^{14}C_{12} = \frac{14 \times 13}{2} = 91$$

22. $\sum_{r=0}^{n} r \cdot {}^nC_r + 2\sum_{r=0}^{n} \frac{1}{r+1} \cdot {}^nC_r; \quad n\sum_{r=0}^{n} {}^{n-1}C_{r-1} + \frac{2}{n+1}\sum_{r=0}^{n} {}^{n+1}C_{r+1}$

$\Rightarrow \quad n \cdot 2^{n-1} + \frac{2}{n+1} \cdot (2^{n+1} - 1)$

Exercise-2 : One or More than One Answer is/are Correct

1. $N = {}^{20}C_7 - {}^{20}C_8 + {}^{20}C_9 - {}^{20}C_{10} + \ldots\ldots - {}^{20}C_{20}$

$= ({}^{20}C_7 + {}^{20}C_9 + {}^{20}C_{11} + \ldots\ldots + {}^{20}C_{19}) - ({}^{20}C_8 + {}^{20}C_{10} + \ldots\ldots + {}^{20}C_{20})$

$= ({}^{20}C_{20} + {}^{20}C_2 + {}^{20}C_4 + {}^{20}C_6) - ({}^{20}C_1 + {}^{20}C_3 + {}^{20}C_5)$

2. For B and D put $x = 1, -1$

For A differentiate with respect to x then put $x = 0$

For C replace x with $\frac{1}{x}$

3. $\sum_{r=0}^{4} (-1)^r \; {}^{16}C_r = {}^{16}C_0 - {}^{16}C_1 + {}^{16}C_2 - {}^{16}C_3 + {}^{16}C_4 = 1365$

4. $2 \times \frac{1}{2} \times {}^nC_1 = 1 + \frac{1}{2^2} \times {}^nC_2 \Rightarrow n = 8, 1$

$$T_{r+1} = {}^8C_r\left(\frac{1}{2}\right)^r x^{\frac{16-3r}{4}} \Rightarrow r = 0, 4, 8$$

5. LHS $= (1 + 2x^2 + x^4)(1 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots)$

RHS $= a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots$

Comparing the coefficients of $x, x^2, x^3, \ldots\ldots$

Now, $\quad 2a_2 = a_1 + a_3$

$$2({}^nC_2 + 2) = {}^nC_1 + ({}^nC_3 + 2\,{}^nC_1)$$

$$2\frac{n(n-1)}{2} + 4 = 3n + \frac{n(n-1)(n-2)}{6}$$

or $\quad n^3 - 9n^2 + 26n - 24 = 0$

$\therefore \quad (n-2)(n^2 - 7n + 12) = 0 \qquad (\because 8 + 52 = 36 + 24)$

or $\quad (n-2)(n-3)(n-4) = 0$

$\therefore \quad n = 2, 3, 4$

6. $\displaystyle\sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} {}^nC_i \cdot {}^nC_j \cdot {}^nC_k = \left(\sum_{i=0}^{n} {}^nC_i\right)\left(\sum_{j=0}^{n} {}^nC_j\right)\left(\sum_{k=0}^{n} {}^nC_k\right) = 2^{3n}$

7. $({}^{100}C_6 + {}^{100}C_7) + 3({}^{100}C_7 + {}^{100}C_8) + 3({}^{100}C_8 + {}^{100}C_9) + ({}^{100}C_9 + {}^{100}C_{10})$

$$= {}^{101}C_7 + 3\,{}^{101}C_8 + 3\,{}^{101}C_9 + {}^{101}C_{10}$$

$({}^{101}C_7 + {}^{101}C_8) + 2({}^{101}C_8 + {}^{101}C_9) + ({}^{101}C_9 + {}^{101}C_{10}) = {}^{102}C_8 + 2\cdot{}^{102}C_9 + {}^{102}C_{10}$

$$= ({}^{102}C_8 + {}^{102}C_9) + ({}^{102}C_9 + {}^{102}C_{10}) = {}^{103}C_9 + {}^{103}C_{10} = {}^{104}C_{10}$$

8. $\dfrac{{}^{15}C_{2r}}{{}^{15}C_{2r+1}} > \dfrac{1}{2} \Rightarrow \dfrac{2r+1}{15-2r} > \dfrac{1}{2} \Rightarrow \dfrac{6r-13}{2r-15} < 0 \Rightarrow \dfrac{13}{6} < r < \dfrac{15}{2}$

9. $f(x) = 1 + x^{111} + x^{222} + \ldots + x^{999}$

if $f(x)$ is divided by $x + 1$, then remainder $f(-1) = 0$

if $f(x)$ is divided by $x - 1$, then remainder $f(1) = 10$

$f(x) = (1 + x^{222} + x^{444} + x^{666} + x^{888}) + x^{111}(1 + x^{222} + x^{444} + x^{666} + x^{888})$

$\quad = (1 + x^{111})(1 + x^{222} + x^{444} + x^{666} + x^{888})$

Exercise-3 : Matching Type Problems

2. (B) $P = \displaystyle\sum_{r=0}^{n} {}^nC_r = 2^n$

$Q = \displaystyle\sum_{r=0}^{m} {}^mC_r(15)^r = (1 + 15)^m = 16^m$

$\Rightarrow \qquad n = 4m$

(C) $1 + 6 + 120 + 56K$

Reminder $= 15$

3. (A) $\dfrac{a^2+b^2+ab}{a+b}=\dfrac{(a-b)(a^2+b^2+ab)}{(a-b)(a+b)}=\dfrac{a^3-b^3}{a^2-b^2}$

$$\frac{4+\sqrt{3}}{\sqrt{3}+1}+\frac{8+\sqrt{15}}{\sqrt{5}+\sqrt{3}}+\frac{12+\sqrt{35}}{\sqrt{7}+\sqrt{5}}+\ldots=\frac{1}{2}((\sqrt{169})^3-1^3)=1098$$

(B) $\dfrac{8}{5}(2\cos^2\theta-3\sin\theta)=\dfrac{8}{5}(-2\sin^2\theta-3\sin\theta+2)$

Greatest value $=5$ at $\sin\theta=-\dfrac{3}{4}$ $(\because 4\le\theta\le 6)$

(C) Let $(\sqrt{3}+1)^6=I+f$

and $(\sqrt{3}-1)^6=f'\Rightarrow(\sqrt{3}+1)^6+(\sqrt{3}-1)^6=416=I+1$

$\Rightarrow$ $I=415=1\times 5\times 83$

(D) $(1+x)(1+x^2)\ldots(1+x^{128})=\dfrac{1-x^{256}}{1-x}=\dfrac{1-x^{n+1}}{1-x}$

$\Rightarrow$ $n+1=256$

Exercise-4 : Subjective Type Problems

2. Coefficient of $x^{60}=-6+5+8-6=1$

7. $(1+x)^{3n}={}^{3n}C_0+{}^{3n}C_1x+{}^{3n}C_2x^2+\ldots\ldots+{}^{3n}C_{3n}x^{3n}$

Put $x=1$ $\quad 2^{3n}={}^{3n}C_0+{}^{3n}C_1+{}^{3n}C_2+\ldots\ldots+{}^{3n}C_{3n}$

Put $x=\omega$ $\quad (-\omega^2)^{3n}={}^{3n}C_0+{}^{3n}C_1\omega+{}^{3n}C_2\omega^2+\ldots\ldots+{}^{3n}C_{3n}$

Put $x=\omega^2$ $\quad (-\omega)^{3n}={}^{3n}C_0+{}^{3n}C_1\omega^2+{}^{3n}C_2\omega^4+\ldots\ldots+{}^{3n}C_{3n}$

$$2^{3n}+(-\omega^2)^{3n}+(-\omega)^{3n}=3[{}^{3n}C_0+{}^{3n}C_3+\ldots\ldots+{}^{3n}C_{3n}]$$

10. $\displaystyle\sum_{K=1}^{5}{}^{20}C_{2K-1}=2^{18}\Rightarrow 2^{108}=2^3(2^5)^{21}=8(33-1)^{21}$

Remainder $=-8$ or 3

11. $f(n)={}^nC_0a^{n-1}-{}^nC_1a^{n-2}+\ldots\ldots$

$\Rightarrow$ $f(n)=\dfrac{(a-1)^n}{a}$

$f(2007)+f(2008)=3^7K$

$\Rightarrow$ $\dfrac{3^9+(a-1)3^9}{a}=3^7K\Rightarrow K=9$

13. $(360+1)^{44}-1={}^{44}C_0\cdot(360)^{44}+{}^{44}C_1\cdot(360)^{43}+\ldots+{}^{44}C_{43}\cdot(360)^1$

$=360[{}^{44}C_0\cdot(360)^{43}+{}^{44}C_1\cdot(360)^{42}+\ldots+{}^{44}C_{43}]$

14. $(3^{|x-2|}+(3^{|x-2|-9})^{1/5})^7$

$T_6={}^7C_5\cdot(3^{|x-2|})^2\cdot 3^{|x-2|-9}=567$

$\Rightarrow\quad 3^{3|x-2|-9}=27\Rightarrow|x-2|=4\Rightarrow x=6,-2$

15. $1+\sum_{r=1}^{10}3^r\cdot{}^{10}C_r+\sum_{r=1}^{10}r\cdot{}^{10}C_r$

$1+((1+3)^{10}-{}^{10}C_0)+10\cdot 2^9=4^{10}+5\cdot 2^{10}=2^{10}(4^5+5)$

$\alpha=1,\beta=5$

if α,β lies between the roots of $f(x)=0$

$f(1)<0\cap f(5)<0$

$-k^2<0\cap 16-k^2<0$

16. $S_n={}^nC_0\,{}^nC_1+{}^nC_1\,{}^nC_2+\ldots+{}^nC_{n-1}\cdot{}^nC_n={}^{2n}C_{n-1}$

$S_{n+1}={}^{2n+2}C_n$

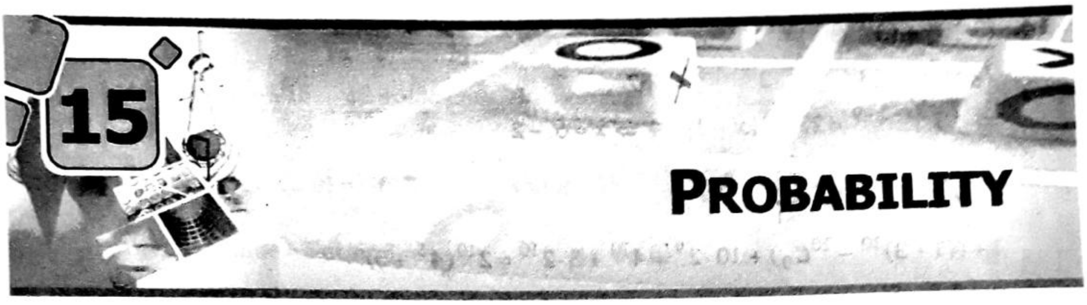
$$\frac{S_{n+1}}{S_n}=\frac{{}^{2n+2}C_n}{{}^{2n}C_{n-1}}=\frac{15}{4}$$

$$\Rightarrow\quad\frac{(2n+2)(2n+1)}{n(n+2)}=\frac{15}{4}$$

$$\Rightarrow\quad n^2-6n+8=0$$

❑❑❑

Chapter 15 – Probability



15

PROBABILITY

Exercise-1 : Single Choice Problems

2. $$f(x)=3\sqrt{x}+4\sqrt{1-x} \qquad \text{[where } x=P(A)\text{]}$$

$$f(x)_{\text{max.}}=5 \text{ at } x=\frac{9}{25}$$

3. $$P(A\cup B)=1-P(\overline{A\cup B})=\frac{5}{6}$$

$$P(A\cap B)=\frac{1}{4},P(A)=\frac{3}{4}$$

$$\Rightarrow \qquad P(B)=P(A\cup B)+P(A\cap B)-P(A)=\frac{1}{3}$$

4. $1-\left(\frac{1}{2}\right)^n=\frac{31}{32}\Rightarrow n=5$

5. Required probability $=\dfrac{3!\times 2}{\dfrac{9!}{3!3!3!}}=\dfrac{1}{140}$

6. Required probability $=\dfrac{\dfrac{3!(3n-3)!}{[(n-1)!]^3}}{\dfrac{(3n)!}{(n!)^3}}$

7. If product of two numbers equal to third number, then possibilities are $(2,3,6)$, $(2,4,8)$, $(2,5,10)$.

Probability $=\dfrac{3}{{}^{10}C_3}=\dfrac{1}{40}$

8. $P=\frac{3}{3}\times\frac{2}{3}\times\frac{2}{3}\times\frac{1}{3}=\frac{4}{27}$

9. Total word $= n = \dfrac{7!}{2!2!}$

$$\text{T}\boxed{\text{ITANIC}} \qquad \text{I}\boxed{\text{TTANIC}} \qquad \text{A}\boxed{\text{TTINIC}}$$

Favourable word $= m = \dfrac{6!}{2!} + \dfrac{6!}{2!} + \dfrac{6!}{2!2!} \Rightarrow P = \dfrac{m}{n} = \dfrac{5}{7}$

10. Probability $= \dfrac{n!}{n^n} = \dfrac{3}{32} = \dfrac{6}{64}$

$$\Rightarrow \quad \frac{(n-1)!}{n^{n-1}} = \frac{1\cdot 2\cdot 3}{4^3} \Rightarrow n = 4$$

11. Total case $= n = 9 \times 10^3$

Favourable case $= m = (9 \times 10^3) - 6^4$

$$P = 1 - \frac{6^4}{9 \times 10^3} = \frac{107}{125}$$

12. Total case $= n = 6!$

Favourable case $= m = (3! \times 2!) + (2! \times 2!) = 16$

Probability $= \dfrac{16}{6!} = \dfrac{1}{45}$

13. E_1 "No card is king from removed cards"

E_2 "1 card is king from removed cards"

E_3 "2 card is king from removed cards"

E_4 "3 card is king from removed cards"

E_5 "4 card is king from removed cards"

$F = 3$ cards are drawn from pack those are kings.

$$P(F) = \sum_{i=1}^{S} P(E_i)\cdot P\left(\frac{F}{E_i}\right) = \frac{{}^{48}C_{26}}{{}^{52}C_{26}} \cdot \frac{{}^{4}C_{3}}{{}^{26}C_{3}} + \frac{{}^{48}C_{25}\cdot {}^{4}C_{1}}{{}^{52}C_{26}} \cdot \frac{{}^{3}C_{3}}{{}^{26}C_{3}} + 0 + 0 + 0$$

$$= \frac{4}{{}^{52}C_{26}\cdot {}^{26}C_{3}}\left({}^{48}C_{26} + {}^{48}C_{25}\right) = \frac{4 \times {}^{49}C_{26}}{{}^{52}C_{26}\cdot {}^{26}C_{3}}$$

$$= \frac{1}{(13)(17)(25)}$$

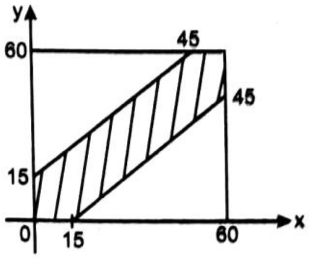
14. $\dfrac{{}^{3}C_{2}\cdot {}^{10}C_{4}}{{}^{13}C_{6}} \times \dfrac{1}{7} = \dfrac{15}{286}$

15. Let f be function from $\{1, 2, \ldots, 10\}$ to itself total functions possible is 10^{10}. The number of one-one onto functions possible is 10!.

Hence, the probability of selected function to be one-one onto is $\dfrac{10!}{10^{10}} = \dfrac{9!}{10^9}$.

16. Let the friends come to the restaurant at 5 h x min and 5 h y min, respectively, where $x, y \in [0, 60]$.

Hence, the sample space consists of all points (x, y) lying in 60×60 square as shown above and for favourable cases, $|x - y| \le 15$, that is $-15 \le x - y \le 15$ which is shown by shaded region in the graph shown below :



Hence, the probability that they will meet is given by :

$$1 - \frac{2 \times \frac{1}{2} \times 45 \times 45}{60 \times 60} = 1 - \left(\frac{3}{4}\right)^2 = \frac{7}{16}$$

17. Total ways = ${}^{91}C_3$

Favourable ways = (Common ratio is 2) + (Common ratio is 3) = 16 + 2 = 18

Exercise-2 : One or More than One Answer is/are Correct

1. Probability $= \dfrac{{}^4C_3 \times {}^{11}C_5}{{}^{15}C_8} \times \dfrac{1}{7} = \dfrac{8}{195}$

2. Probability $= \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{6}\right)\left(1 - \frac{1}{8}\right) \ldots\ldots \left(1 - \frac{1}{2012}\right)$

$$= \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \frac{7}{8} \times \ldots\ldots \times \frac{2011}{2012} = \frac{2012!}{2^{2012}(1006!)^2}$$

3. We have $P(E_i) = \dfrac{2}{4} = \dfrac{1}{2}$ or $i = 1, 2, 3$.

Also for $i \ne j$, $P(E_i \cap E_j) = \dfrac{1}{4} = P(E_j)P(E_i)$. Therefore, E_i and E_j are independent for $i \ne j$.

Also, $P(E_1 \cap E_2 \cap E_3) = \dfrac{1}{4} \ne P(E_1)P(E_2)P(E_3)$

$\therefore$ E_1, E_2, E_3 are not independent.

4. Max. $(P(A \cap B)) = P(A) = \dfrac{3}{5}$

$$\text{Min. } (P(A\cap B)) = P(A)+P(B)-1=\frac{4}{15}$$

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)=\frac{19}{15}-P(A\cap B)$$

$$P(A\cap \bar{B})=P(A)-P(A\cap B)=\frac{3}{5}-P(A\cap B)$$

$$P\left(\frac{\bar{A}}{B}\right)=\frac{P(\bar{A}\cap B)}{P(B)}=\frac{P(B)-P(A\cap B)}{P(B)}$$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 2

1. $P(E_1)=\frac{1}{10}\times 1+\frac{2}{10}\times\frac{1}{2}+\frac{3}{10}\times\frac{1}{3}+\frac{4}{10}\times\frac{1}{4}=\frac{4}{10}=\frac{2}{5}$

2. $$P\left(\frac{B_3}{E_2}\right)=\frac{P(B_3\cap E_2)}{P(E_2)}=\frac{\frac{3}{10}\times\frac{1}{3}}{\frac{2}{10}\times\frac{1}{2}+\frac{3}{10}\times\frac{1}{3}+\frac{4}{10}\times\frac{1}{4}}=\frac{1}{3}$$

Paragraph for Question Nos. 3 to 5

3. Mr. A's 3 digit number is always greater than Mr. B's 3 digit numbers then A should always pick digit 9.

$$\text{Probability}=\frac{{}^8C_3\times{}^8C_2}{{}^8C_3\times{}^9C_3}=\frac{1}{3}$$

4. $$\text{Probability}=\frac{{}^8C_3\times 1}{{}^9C_3\times{}^8C_3}=\frac{1}{{}^9C_3}=\frac{1}{84}$$

5. $P(E)=A$ picks 9 or A does not pick 9 and his number is greater than B

$$=\frac{1}{3}+\frac{2}{3}\cdot\frac{1}{2}\left(1-\frac{{}^8C_3}{{}^8C_3}\cdot\frac{1}{{}^8C_3}\right)=\frac{37}{56}$$

Paragraph for Question Nos. 6 to 7

6. Let a_n = number of ways of outcomes of n tosses when no 2 consecutive heads occur

$a_n=a_{n-2}+a_{n-1}$

Also, $a_1=2$ (H or T)

$a_2=3$ (TT or HT or TH)

$\therefore$ $a_3=5, a_4=8$ ……

$a_{10}=144$

$\therefore$ Probability $= \dfrac{144}{2^{10}}$

7. [HT HT HTH] T, T, T

Number of ways of arranging $= \dfrac{4!}{3!} = 4$

$$\text{Probability} = \frac{4}{2^{10}}$$

Paragraph for Question Nos. 8 to 10

8. $6n > 2^n, n \in N$

$\therefore \quad n = 1, 2, 3, 4$

9. $$\frac{4}{6} \times \left(\frac{\text{Number of solutions of } x + y > 4, 1 \le x, y \le 6}{36}\right)$$

$$\times \left(\frac{\text{Number of solutions of } x + y + z > 8, 1 \le x, y, z \le 6}{6^3}\right)$$

$$= \frac{4}{6} \times \frac{30}{36} \times \frac{160}{216} = \frac{100}{243}$$

10. Probability $= \dfrac{4}{6} \times \dfrac{30}{36} \times \left(1 - \dfrac{160}{216}\right) = \dfrac{4}{6} \times \dfrac{30}{36} \times \dfrac{56}{216} = \dfrac{35}{243}$

Paragraph for Question Nos. 11 to 12

11. Let p_1 be the probability of being an answer correct from section 1. Then $p_1 = 1/5$. Let p_2 be the probability of being an answer correct from section 2. Then $p_2 = 1/15$.

Hence, the required probability is $\dfrac{1}{5} \times \dfrac{1}{15} = \dfrac{1}{75}$

12. Scoring 10 marks from four questions can be done in $3 + 3 + 3 + 1 = 10$ ways so as to answer 3 questions from section 2 and 1 question from section 1 correctly.

Hence, the required probability is $\dfrac{{}^{10}C_3 \, {}^{10}C_1}{{}^{20}C_4} \dfrac{1}{5} \left(\dfrac{1}{15}\right)^3$.

Exercise-5 : Subjective Type Problems

1. $\left(\dfrac{2}{3} + \dfrac{1}{3} \cdot \dfrac{2}{3} + \dfrac{1}{3} \cdot \dfrac{1}{3} \cdot \dfrac{2}{3}\right)\left(\dfrac{2}{3} + \dfrac{1}{3} \cdot \dfrac{2}{3}\right)\dfrac{2}{3} = \dfrac{416}{729}$

5. Probability $= \dfrac{{}^6C_5 + {}^7C_4 + {}^8C_3 + {}^9C_2 + {}^{10}C_1 + 1}{2^{10}} = \dfrac{9}{64}$

6. $p=\frac{{}^3C_1}{{}^7C_1}=\frac{3}{7}$

7. Total ways $=\frac{6!}{2!2!2!3!}\times 3!=90$

Favourable cases $=90-[3!+{}^3C_1\times{}^3C_1\times 2\times 2]=48$

$$\Rightarrow \qquad p=\frac{48}{90}=\frac{8}{15}$$

9. $E_1\rightarrow$ be the event of both getting the correct answer

$E_2\rightarrow$ both getting wrong answers.

$E\rightarrow$ both obtaining same answer.

$$P(E_1)=\frac{1}{8}\frac{1}{12}=\frac{1}{96},\quad P(E_2)=\left(1-\frac{1}{8}\right)\left(1-\frac{1}{12}\right)=\frac{77}{96}$$

$$P\left(\frac{E}{E_1}\right)=1;\quad P\left(\frac{E}{E_2}\right)=\frac{1}{1001}$$

$$P\left(\frac{E_1}{E}\right)=\frac{1\cdot\frac{1}{96}}{1\cdot\frac{1}{96}+\frac{1}{1001}\cdot\frac{77}{96}}=\frac{13}{14}$$

10. Total ways $={}^9C_7\times 7!$

Favourable ways $\Rightarrow {}^9C_7\times 7!-({}^7C_3\times 3!)\times({}^6C_4\times 4!)$

$$P(E)=1-\frac{({}^7C_3\times 3!)\times({}^6C_4\times 4!)}{{}^9C_7\times 7!}=1-\frac{15}{36}=\frac{7}{12}$$

11. $\frac{1}{2}\left\{\frac{1}{2}\times\frac{1}{2}+2\times\frac{1}{4}\times\frac{1}{4}\right\}+\frac{1}{4}\times\left\{\left(\frac{1}{4}\times\frac{1}{4}\times\frac{1}{2}\right)^3\right\}=\frac{27}{128}$

12. $1^{st}2^{nd}$ $\quad\frac{1}{4}\times\frac{1}{6}\quad a$

$2^{nd}1^{st}$ $\quad\frac{1}{4}\times\frac{1}{6}\quad b$

$1^{st}1^{st}$ $\quad\frac{1}{4}\times\frac{1}{36}\quad c$

$2^{nd}2^{nd}$ $\quad\frac{1}{4}\times\frac{1}{36}\quad d$

$\frac{c+d}{a+b+c+d}=\frac{2}{5}$

* * *

Chapter 16 – Logarithms

16

LOGARITHMS

Exercise-1 : Single Choice Problems

1. $\log_{10} x = A \qquad x > 0$

$\log_{10}(x-2) = B, \quad x - 2 > 0 \Rightarrow x > 2$

$\Rightarrow \qquad A^2 - 3AB + 2B^2 < 0$

$\Rightarrow \qquad (A - 2B)(A - B) < 0$

$\Rightarrow \qquad (\log x - 2\log(x-2))(\log x - \log(x-2)) < 0$

Case-I : $\qquad \log x - 2\log(x-2) < 0$

and $\qquad \log x - \log(x-2) > 0 \qquad \ldots(1)$

Case-II : $\qquad \log x - 2\log(x-2) > 0$

and $\qquad \log x - \log(x-2) < 0 \qquad \ldots(2)$

From (1) & (2), $\qquad x \in (4, \infty)$

2. $\qquad (\log_e x)^2 = (\log_e x) - (\log_e x)^2 + 1 \qquad (\log_e x > 0)$

$2(\log_e x)^2 - \log_e x - 1 = 0$

$(2\log_e x + 1)(\log_e x - 1) = 0$

$\log_e x = -\frac{1}{2}$ (not possible)

$\log_e x = 1$

3. $\qquad S = (a^{\log_3 7})^{\log_3 7} + (b^{\log_7 11})^{\log_7 11} + (c^{\log_{11} 25})^{\log_{11} 25}$

$= 27^{\log_3 7} + 49^{\log_7 11} + \sqrt{11}^{\log_{11} 25} = 469$

4. $a^2 - 3a + 3 > \left(x + \frac{1}{x}\right)^0$ and $a^2 - 3a + 3 > 0$

$a^2 - 3a + 2 > 0$

$(a-1)(a-2) > 0 \Rightarrow a \in (-\infty, 1) \cup (2, \infty)$

5. $P = \dfrac{5}{\log_x 120} = \log_{120} x^5; \quad (120)^P = x^5 = 32 \Rightarrow x = 2$

6. $x = \dfrac{z^{1/3}}{2},\ y = \dfrac{z^{1/6}}{5}$

If $xy = z^{3/2}$; $\dfrac{z^{1/3}}{2} \cdot \dfrac{z^{1/6}}{5} = z^{3/2} \Rightarrow z = \dfrac{1}{10}$

7. $\log_x(\log_3(\log_x y)) = 0 \Rightarrow y = x^3, \log_y 27 = 1 \Rightarrow y = 27$

8. $\log_{10^{-2}} 10^3 + \log_{10^{-1}} 10^{-4} = \dfrac{-3}{2} + 4 = \dfrac{5}{2}$

9. $a = \dfrac{3}{1 + 2\log_3 2} \Rightarrow \log_3 2 = \dfrac{3-a}{2a}$; $\log_6 16 = \dfrac{4\log_3 2}{1 + \log_3 2}$

10. $\log_2(\log_2(\log_3 x)) = 0 \quad \Rightarrow x = 9$

$\log_2(\log_3(\log_2 y)) = 0 \quad \Rightarrow y = 8$

11. Let $\log_3 a = x$, $\log_3 b = y$; $\dfrac{x}{3} + \dfrac{y}{2} = \dfrac{7}{2}$ and $\dfrac{x}{2} + \dfrac{y}{3} = \dfrac{2}{3}$

12. $a = \log_2 5$; $b = \log_5 8$; $c = \log_8 11$; $d = \log_{11} 14$

$$2^{abcd} = 2^{\log_2 14} = 14$$

14. $\dfrac{\log_8 17}{\log_9 23} = \dfrac{\log_{2\sqrt{2}} 17}{\log_3 23}$

15. **Case-I :** $2x - 3 > 1$, $3x - 4 > 1$, $x > \dfrac{5}{3} \Rightarrow x > 2$

Case-II : $0 < 2x - 3 < 1$, $0 < 3x - 4 < 1$, $x < \dfrac{5}{3} \Rightarrow \dfrac{3}{2} < x < \dfrac{5}{3}$

16. $p \le \log_{10} N < p + 1 \quad \Rightarrow P = 10^{p+1} - 10^p$

$-q \le \log_{10} 1/N < -q + 1 \quad \Rightarrow Q = 10^q - 10^{q-1}$

17. $n + 1 =$ number of digits $= 1 +$ characteristic

18. $\log_{10}(0.15)^{20} = 20(\log_{10} 15 - 2) = -16.478$

19. $\log_2(\log_4(\log_{10} 10^{16})) = \log_2(\log_4 16) = 1$

20. $2\log x - \log(2x - 75) = 2$

$$\frac{x^2}{2x - 75} = 100 \Rightarrow x^2 - 200x + 7500 = 0$$

21. $x^{\log_x a \cdot \log_a y \cdot \log_y z} = x^{\log_x z} = z$

22. $x^{x\sqrt{x}} = x^{3x/2}$

$x \ne 0, 1 \qquad x\sqrt{x} = \dfrac{3}{2}x \Rightarrow x = \dfrac{9}{4}$

If $x = 1$, then it also satisfy.

23. $(\log_3 x)^2 = 2\log_3 x$

$\Rightarrow \quad \log_3 x = 0 \quad$ or $\quad \log_3 x = 2$

$x = 1 \quad$ or $\quad x = 9$

24. $\log_{10} x + \log_{10} y = 2 \Rightarrow xy = 100 \quad$ ...(1)

$x - y = 15 \quad$ ...(2)

$\Rightarrow \quad x = 20, \quad y = 5$

25. $$\left(2^{x+\frac{1}{3}\left(2x-\frac{3}{x}\right)}\right)^{1/2} = 2^{\frac{7}{3}}$$

$\Rightarrow \quad x + \frac{2}{3}x - \frac{1}{x} = \frac{14}{3} \Rightarrow 5x^2 - 14x - 3 = 0$

26. $$25^{(2x-x^2+1)} + 9^{(2x-x^2+1)} = 34\frac{3^{2x-x^2+1}}{3}\cdot\frac{5^{2x-x^2+1}}{5}$$

Let $3^{2x-x^2+1} = a$ and $5^{2x-x^2+1} = b$

$$a^2 + b^2 = \frac{34}{15}ab$$

$$15a^2 - 34ab + 15b^2 = 0 \Rightarrow (3a-5b)(5a-3b) = 0$$

Case-1 : if $\frac{a}{b} = \frac{5}{3}$

$\Rightarrow \quad \left(\frac{3}{5}\right)^{2x-x^2+1} = \frac{5}{3}$

$\Rightarrow \quad 2x - x^2 + 1 = -1 \Rightarrow x^2 - 2x - 2 = 0$

Sum of two values of $x = 2$

Case-2 : if $\frac{a}{b} = \frac{3}{5}$

$$\left(\frac{3}{5}\right)^{2x-x^2+1} = \frac{3}{5}$$

$\Rightarrow \quad 2x - x^2 + 1 = 1 \Rightarrow x = 0$ and 2

Sum of all values of x is 4.

27. $a^x = b^y = c^z = d^w$

$\Rightarrow \quad b = a^{x/y},\ c = a^{x/z},\ d = a^{x/w}$

$$\log_a(bcd) = \log_a a^{\left(\frac{x}{y}+\frac{x}{z}+\frac{x}{w}\right)} = \frac{x}{y} + \frac{x}{z} + \frac{x}{w} = x\left(\frac{1}{y} + \frac{1}{z} + \frac{1}{w}\right)$$

28. $x = \dfrac{4}{(\sqrt{5}+1)(\sqrt[4]{5}+1)(\sqrt[8]{5}+1)(\sqrt[16]{5}+1)}$

Multiply and divide by $(1-\sqrt[16]{5})$ then

$$x = -1 + \sqrt[16]{5}$$

$$(x+1)^{48} = 5^3 = 125$$

29. $$\log_x \log_{18}(\sqrt{2}+\sqrt{8}) = \frac{1}{3}$$

$$\log_x \log_{(3\sqrt{2})^2} 3\sqrt{2} = \frac{1}{3}$$

$$\log_x\left(\frac{1}{2}\right) = \frac{1}{3} \Rightarrow x = \frac{1}{8}$$

30. $f(n) = \dfrac{1}{3}\log_2 n$ if $\log_8 n$ is integer

$= 0$ otherwise

$$\sum_{n=1}^{2011} f(n) = \log_8 2^3 + \log_8 2^6 + \log_8 2^9 = 1+2+3 = 6$$

32. $\log_{0.3}(x-1) < \log_{(0.3)^2}(x-1) \Rightarrow (x-1)^2 > x-1$

$\Rightarrow (x-1)(x-2) > 0$

Also, for log to be defined $(x-1) > 0$

$$x \in (2, \infty)$$

33. $$\sqrt{7^{2x^2-5x-6}} = (49)^{3\log_2\sqrt{2}} = 7^3$$

$\Rightarrow$ $2x^2 - 5x - 6 = 6$

$2x^2 - 5x - 12 = 0$

$\Rightarrow$ $(2x+3)(x-4) = 0$

34. $(\log_2 x)^4 + 16(\log_2 x)^2 \log_2 \dfrac{16}{x}$

$\Rightarrow t^4 + 16t^2(4-t)$ (where $\log_2 x = t$)

$\Rightarrow t^2(t^2 + 64 - 16t)$

$\Rightarrow t^2(t-8)^2$

Since $1 \le x \le 256 \Rightarrow 0 \le t \le 8$

$\Rightarrow$ Maximum of $(t-8)^2 t^2$ lies at $t = 4$.

Hence, maximum $(4-8)^2 \cdot 4^2 = 256$

37. $\because \quad \lambda > 0$

$\therefore \quad \log_{16} x = \dfrac{1 \pm \sqrt{(1 - 4\log_{16}\lambda)}}{2}$

$\because$ The given equation will have exactly one solution, if

$1 - 4\log_{16}\lambda = 0 \quad \text{or} \quad \log_{16}\lambda = \dfrac{1}{4} = 4^{-1}$

$\therefore \quad \lambda = (16)^{4^{-1}} = (2^4)^{1/4} = 2, -2, 2i, -2i$, where $i = \sqrt{-1}$

But λ is real and positive.

$\therefore \quad \lambda = 2$

Number of real values $= 1$

38. Let x be the rational number, then according to question,

$$x = 50 \times \log_{10} x$$

By trial $x = 100$

39. $\because \quad x = \log_5(1000) = \log_5(5^3 \times 8) = 3 + \log_5 8$

and $y = \log_7(2058) = \log_7(7^3 \times 6) = 3 + \log_7 6$

$\Rightarrow \quad x - y = \log_5 8 - \log_7 6 > 0 \qquad \left(\begin{array}{l} \because \log_5 8 > 1, \log_7 6 < 1 \\ \therefore \log_5 8 - \log_7 6 > 0 \end{array}\right)$

$\therefore \quad x > y$

40. $7\log\left(\dfrac{2^4}{5 \times 3}\right) + 5\log\left(\dfrac{5^2}{2^3 \times 3}\right) + 3\log\left(\dfrac{3^4}{2^4 \times 5}\right)$

$= 7\{4\log 2 - \log 5 - \log 3\} + 5\{2\log 5 - 3\log 2 - \log 4\} + 3\{4\log 3 - 4\log 2 - \log 5\}$

$= \log 2$

41. $\log_{10}\{\tan 1^\circ \tan 2^\circ \tan 3^\circ \ldots \tan 45^\circ \ldots \tan 87^\circ \tan 88^\circ \tan 89^\circ\}$

$= \log_{10}\{\tan 1^\circ \tan 2^\circ \tan 3^\circ \ldots \tan 45^\circ \ldots \cot 3^\circ \cot 2^\circ \cot 1^\circ\}$

$= \log_{10} 1 = 0$

42. $\log_7 \log_7 7^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}} = \log_7\left(\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8}\right) = \log_7\left(\dfrac{7}{8}\right)$

$= 1 - \log_7 8 = 1 - 3\log_7 2$

43. $(4)^{\log_{3^2} 3} + (9)^{\log_2 2^2} = (10)^{\log_x 83}$

$\Rightarrow \quad (4)^{1/2} + 9^2 = (10)^{\log_x 83}$

$\Rightarrow \quad (83)^1 = (83)^{\log_x 10}$

$\therefore \quad 1 = \log_x 10 \Rightarrow x = 10$

44. $(10^{\log_{10} x})^{\log_{10}\left(\frac{y}{z}\right)} (10^{\log_{10} y})^{\log_{10}\left(\frac{z}{x}\right)} (10^{\log_{10} z})^{\log_{10}\left(\frac{x}{y}\right)}$

45. $\because \quad \log_x 2 \log_{2x} 2 = \log_{4x} 2$

$\therefore \quad x > 0, 2x > 0$ and $4x > 0$ and $x \neq 1, 2x \neq 1, 4x \neq 1$

$\Rightarrow \quad x > 0$ and $x \neq 1, \frac{1}{2}, \frac{1}{4}$

Then, $\quad \frac{1}{\log_2 x} \cdot \frac{1}{\log_2 2x} = \frac{1}{\log_2 4x}$

$\Rightarrow \quad \log_2 x \cdot \log_2 2x = \log_2 4x$

$\Rightarrow \quad \log_2 x \cdot (1 + \log_2 x) = (2 + \log_2 x)$

$\Rightarrow \quad (\log_2 x)^2 = 2$

$\Rightarrow \quad \log_2 x = \pm\sqrt{2}$

$\therefore \quad x = 2^{\pm\sqrt{2}}$

$\therefore \quad x = \{2^{-\sqrt{2}}, 2^{\sqrt{2}}\}$

46. $\because \quad 2\log_{10} x - \log_x(0.01) = 2\log_{10} x - \log_x(10^{-2})$

$= 2(\log_{10} x + \log_x 10)$ $\quad$ $(\because x > 0$ and $x \neq 1)$

$= 2\left(\frac{\log_e x}{\log_e 10} + \frac{\log_e 10}{\log_e x}\right) \geq 2 \cdot 2$ $\quad$ $(\because \text{AM} \geq \text{GM})$

$= 4$

47. Let $\sqrt{\log_2 x} = a$

$a^2 - 2a + 1 \quad \Rightarrow a = 1$

if $\quad \sqrt{\log_2 x} = 1 \quad \Rightarrow x = 2$

48. $\log_e(e^2 x^{\ln x}) = \log_e x^3$

$2 + (\ln x)^2 = 3\ln x$

Let $\ln x = a$

$a^2 - 3a + 2 = 0 \quad \Rightarrow (a-2)(a-1) = 0$

$\Rightarrow x_1 = e^2, x_2 = e$

49. $M = \text{antilog}_{32} 0.6 = (32)^{0.6} = 2^3 = 8$

$N = 49^1 \cdot 49^{-\log_7 2} + 5^{-\log_5 4}$

$= \frac{49}{4} + \frac{1}{4} = \frac{25}{2}$

50. $\log_2(\log_2(\log_3 x)) = 0 \quad \Rightarrow x = 9$

$\log_3(\log_3(\log_2 y)) = 0 \quad \Rightarrow y = 8$

51. $|\log_{1/2} 10 + |\log_4 625 - \log_2 5| = |\log_{1/2} 10 + \log_2 5| = 1$

52. $\log_3 2 = \dfrac{\log_5 2}{\log_5 3} = \dfrac{\left(\dfrac{1}{2a}\right)}{b - \dfrac{1}{2a}} = \dfrac{1}{2ab - 1}$

55. $(x-3)^2 = 9 \Rightarrow x = 6$

57. $\log_a\left[\left(\dfrac{16}{15}\right)^7 \cdot \left(\dfrac{25}{24}\right)^5 \cdot \left(\dfrac{81}{80}\right)^3\right] = 8$

$\Rightarrow \qquad \log_a 2 = 8 \qquad \Rightarrow \; a = 2^{1/8}$

58. $\log_{2^3}(2^7) - \log_{3^2}(3^{-1/2}) = \dfrac{7}{3} + \dfrac{1}{4} = \dfrac{31}{12}$

59. $\left(\dfrac{1}{\sqrt{27}}\right)^2 \cdot \left(\dfrac{1}{\sqrt{27}}\right)^{-\left(\frac{\log_5 16}{2\log_5 9}\right)} = \left(\dfrac{1}{27}\right)\left(\dfrac{1}{\sqrt{27}}\right)^{-\log_3 2}$

$= \left(\dfrac{1}{27}\right) \cdot 2^{-\log_3 \frac{1}{\sqrt{27}}} = \dfrac{2\sqrt{2}}{27}$

60. $\log_2 \dfrac{(x-1)(x+2)}{3x-1} = \log_2 4 \qquad \Rightarrow \dfrac{(x-1)(x+2)}{3x-1} = 4$

$\Rightarrow \; x^2 - 11x + 2 = 0$

61. $\log_{100} 10 = \dfrac{1}{2}$

$\log_2(\log_4 2) = \log_2 1/2 = -1$

$\log_4[\log_2(256)^2]^2 = \log_4 16^2 = 4$

$\log_4 8 = \log_{2^2} 2^3 = \dfrac{3}{2}$

62. $\lambda = \log_5(\log_5 3) \Rightarrow 5^\lambda = \log_5 3$

$3^{k+5^{-\lambda}} = 3^k \cdot 3^{5^{-\lambda}} = 3^k \cdot 3^{\log_3 5} = 5 \cdot 3^k$

63. $\log_{10} b^4 = 2\pi \cdot \log_{10} a^2$

$\dfrac{\log_{10} b}{\log_{10} a} = \log_a b = \pi$

64. $2^x = 3^y = 6^{-z} = k$ (let)

$x = \log_2 k,\; y = \log_3 k,\; z = -\log_6 k$

$\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = \log_k 2 + \log_k 3 - \log_k 6 = 0$

65. $(\sqrt{2} - 1)^3 = 5\sqrt{2} - 7$

66. $1+\log_a b=\frac{1}{4} \Rightarrow \log_a b=-\frac{3}{4} \Rightarrow \frac{\frac{1}{3}-\frac{1}{2}\log_a b}{1+\log_a b}=\frac{17}{6}$

68. Let $\log_y x=t$

$5t^2-26t+5=0 \Rightarrow (5t-1)(t-5)=0$

Either $x=y^5$ or $y=x^5$

69. $1-\frac{1}{\log_3 x}=\frac{1}{\log_3 x-1} \Rightarrow (\log_3 x-1)^2=\log_3 x \Rightarrow (\log_3 x)^2-3\log_3 x+1=0$

70. $\log_2 x+\frac{1}{2}\log_2 y+\frac{1}{2}\log_2 z=2 \Rightarrow \quad x\sqrt{y}\sqrt{z}=4$

$\log_3 y+\frac{1}{2}\log_3 x+\frac{1}{2}\log_3 z=2 \Rightarrow \sqrt{x}\cdot y\cdot\sqrt{z}=9$

$\log_4 z+\frac{1}{2}\log_4 x+\frac{1}{2}\log_4 y=2 \Rightarrow \sqrt{x}\cdot\sqrt{y}\cdot z=16 \Rightarrow xyz=24$

71. $\left(\frac{1}{49}\right)\cdot 2^{\log_7^{1/49}}+7^{-\log_{1/5}^{5}}$

$\frac{1}{49}\times\frac{1}{4}+7$

72. $\log_2(3-x)-\log_2\frac{1}{\sqrt{2}}+\log_2(5-x)=\frac{1}{2}+\log_2(x+7)$

$\Rightarrow \quad \log_2(3-x)(5-x)=\log_2(x+7)$

$\Rightarrow \qquad x^2-9x+8=0 \quad \Rightarrow x=8$

73. $\log_5 x=\log_x 5 \Rightarrow x=5, \frac{1}{5}$

74. $|x-1|^{\log_3 x^2-2\log_x 9}=(x-7)^7$

either $x=2$ or $\log_3 x^2-2\log_x 9=7$

$(\log_3 x-4)(2\log_3 x+1)=0$

75. $9^{x-1}+7=4(3^{x-1}+1)$

Let $3^x=t$

$\frac{t^2}{9}+7=4\left(\frac{t}{3}+1\right) \quad \Rightarrow t^2-12t+27=0$

$(t-3)(t-9)=0$

76. If $\alpha>1$

$\log_\alpha 10>\log_\alpha 3>\log_\alpha e>\log_\alpha 2$

$\Rightarrow \log_{10}\alpha<\log_3\alpha<\log_e\alpha<\log_2\alpha$

78. $\sum_{r=1}^{4} \log_4 2^r = \sum_{r=1}^{4} \frac{r}{2} = 5$

79. $\log_3 2 + \log_3 5 = \log_3 10$

$\log_3 9 < \log_3 10 < \log_3 27$

80. $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$$\frac{k}{2}\left(a^3 + \frac{1}{a^3}\right) = \frac{3}{2}\left(a + \frac{1}{a}\right) - \frac{4}{8}\left(a + \frac{1}{a}\right)^3$$

$$= \frac{-1}{2}\left(a^3 + \frac{1}{a^3}\right)$$

$\Rightarrow \qquad k = -1$

81. $2x - 3 > 0 \cap x^2 - 5x - 6 > 0 \cap 2x - 3 \neq 1$

$x > \frac{3}{2} \cap (x-6)(x+1) > 0 \cap x \neq 2$

$\Rightarrow \ (6, \infty)$

Exercise-2 : One or More than One Answer is/are Correct

1.

$$6(\log x)^2 + \log x - 1 = 0$$

$$(3\log x - 1)(2\log x + 1) = 0$$

$$x = 10^{1/3} \quad \text{or} \quad x = 10^{-1/2}$$

3. $3(\log_{10} 2)x^2 - (1 - \log_{10} 2)x = 2\log_{10} 2 - x$

$\log_{10} 2(3x^2 + x - 2) = 0$

$\log_{10} 2(x+1)(3x-2) = 0$

Roots of this eq. are $x = -1, \frac{2}{3}$

Sum of coeff. $= 2\log_{10} 2$ (irrational)

Discriminant $= b^2 - 4ac = 25(\log_{10} 2)^2$ (irrational)

4. $A = \min.(x^2 - 2x + 7) \ \forall\ x \in R \qquad \Rightarrow A = 6$

$B = \min.(x^2 - 2x + 7) \ \forall\ x \in [2, \infty) \qquad \Rightarrow B = 7$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 3

1. If $\alpha_1 = 4$, then $3^4 \le N < 3^5$

If $\alpha_2 = 2$, then $5^2 \le N < 5^3$

$\Rightarrow$ $81 \le N < 125$

3. If $\alpha_1 = 5$, then $3^5 \le N < 3^6$

If $\alpha_2 = 3$, then $5^3 \le N < 5^4$

If $\alpha_3 = 2$, then $7^2 \le N < 7^3$

$\Rightarrow$ $243 \le N < 343$

Paragraph for Question Nos. 4 to 5

Sol. $|x^2 - y^2| = 221$

Paragraph for Question Nos. 6 to 7

Sol. $(1 + 4\log_{p^2}(2p))^2 + (1 + \log_2 p)^2 = (1 + \log_2 4p)^2$

Let $\log_2 p = t$

$$\left(1 + 2\left(\frac{1+t}{t}\right)\right)^2 + (1+t)^2 = (3+t)^2 \Rightarrow t = \log_2 p = 2$$

Exercise-4 : Matching Type Problems

1. (A) $a = 3((\sqrt{7}+1) - (\sqrt{7}-1)) = 6$

$b = \sqrt{1296} = 36$

(B) $a = (\sqrt{3}+1) - (\sqrt{3}-1) = 2$

$b = (3+\sqrt{2}) - (3-\sqrt{2}) = 2\sqrt{2}$

(C) $a = (\sqrt{2}+1)$, $b = (\sqrt{2}-1)$

(D) $a = 2+\sqrt{3}$, $b = 2-\sqrt{3}$

Exercise-5 : Subjective Type Problems

1. $N = 6^{\log_{10} 40} . 6^{2\log_{10} 5} = 6^{\log_{10} 1000} = 6^3 = 216$

2. $\log_b(a^{\log_2 b}) = \log_a(b^{\log_2 b})$

$\Rightarrow \quad \log_b a = \log_a b \quad \Rightarrow \quad a = b$ or $a = \frac{1}{b}$ (not possible)

$\log_a(c-(b-a)^2) = 3 \quad \Rightarrow \quad c = a^3$

$\Rightarrow$ Minimum value of $c = 8$ at $a = 2$

3. $\log_b 729 = 6\log_b 3$

if this is an integer, then $b = 3, 3^2, 3^3, 3^6$

4. **Case-1 :** If $\quad x + \frac{5}{2} > 1 \Rightarrow x > -\frac{3}{2}$

then $\quad (x-5)^2 < (2x-3)^2 \Rightarrow 3x^2 - 2x - 16 > 0 \Rightarrow x \in \left(\frac{8}{3}, \infty\right)$

Case-2 : If $\quad 0 < x + \frac{5}{2} < 1 \Rightarrow -\frac{5}{2} < x < -\frac{3}{2}$

then $\quad (x-5)^2 > (2x-3)^2 \Rightarrow x \in \left(-2, -\frac{3}{2}\right)$

there is no negative integral value of x.

5. $\frac{6}{5}a^{(\log_a x)(\log_{10} a)(\log_a 5)} - 3^{(\log_{10} x-1)} = 9^{\left(\log_{100} x+\frac{1}{2}\right)}$

$$6 \cdot 5^{(\log_{10} x-1)} - 3^{(\log_{10} x-1)} = 3^{(\log_{10} x+1)}$$

$$6 \cdot 5^{(\log_{10} x-1)} = \frac{3^{\log_{10} x}}{3} + 3 \cdot 3^{\log_{10} x}$$

$$6 \cdot 5^{(\log_{10} x-1)} = \frac{10}{3} \cdot 3^{\log_{10} x}$$

$$\left(\frac{5}{3}\right)^{\log_{10} x-2} = 1$$

$$\Rightarrow \quad \log_{10} x - 2 = 0$$

$$\Rightarrow \quad x = 100$$

Integer part of $\log_3 100$ is 4.

6. $$\log_5\left(\frac{a+b}{3}\right) = \frac{\log_5 a + \log_5 b}{2}$$

$$\Rightarrow \quad \log_5\left(\frac{a+b}{3}\right)^2 = \log_5(ab)$$

$$\Rightarrow \quad (a+b)^2 = 9ab \Rightarrow a^2 - 7ab + b^2 = 0$$

$$a^4 + b^4 + 2a^2b^2 = 49a^2b^2$$

$$\Rightarrow \quad \frac{a^4 + b^4}{a^2b^2} = 47$$

8. $\log_{10}\sqrt{1+x}+3\log_{10}\sqrt{1-x}=2+\log_{10}\sqrt{1-x}+\log_{10}\sqrt{1+x}$

$\Rightarrow \qquad \log_{10}\sqrt{1-x}=1$

$\sqrt{1-x}=10 \Rightarrow x=-99$ (not possible)

9. $x^2=1+6\log_4 y$

$y^2-2^x y-2^{2x+1}=0$

$\Rightarrow \qquad y=2^{x+1}$ and $y=-2^x$

if $y=-2^x$ (not possible, because $y>0$)

if $y=2^{x+1}$

$\Rightarrow \qquad \log_2 y=x+1$

$x^2=1+3\log_2 y$

$\Rightarrow \qquad x^2=1+3(x+1)$

$x^2-3x-4=0$

$\Rightarrow \qquad (x-4)(x+1)=0$

$x_1=4$

$\Rightarrow \qquad y_1=2^5=32$

$x_2=-1$

$\Rightarrow \qquad y_2=2^{\circ}=1$

$\log_2|x_1x_2y_1y_2|=\log_2 128=7$

10. $\log_7\log_7\sqrt{7\sqrt{7\sqrt{7}}}=\log_7\log_7(7^{7/8})=\log_7(7/8)=1-3\log_7 2$

$\Rightarrow \qquad a=3$

$\log_{15}\log_{15}\sqrt{15\sqrt{15\sqrt{15\sqrt{15}}}}=\log_{15}\log_{15}(15^{15/16})$

$=\log_{15}\left(\dfrac{15}{16}\right)=1-4\log_{15}2$

$\Rightarrow \qquad b=4$

Then $\qquad a+b=7$

11. $\log_{1+x}(1-y)+\log_{1-y}(1+x)=2 \qquad \left(t+\dfrac{1}{t}=2 \Rightarrow t=1\right)$

$1+x=1-y$

$x=-y$

$\therefore \qquad \log_{1-y}(1+2y)+\log_{1-y}(1-2y)=2$

$$\log_{1-y}(1-4y^2)=2$$

$$1-4y^2=1+y^2-2y$$

$$5y^2-2y=0$$

$$y=0, y=\frac{2}{5}$$

But $y=0$ rejected.

12. $$\log_b n=2$$

$$\log_n(2b)=\log_n 2+\log_n b=2$$

$$\log_n 2+\frac{1}{2}=2$$

$$\log_n 2=\frac{3}{2} \Rightarrow n=2^{2/3}$$

if $$\log_b n=2 \Rightarrow b=n^{1/2}=2^{1/3}$$

$$n\cdot b=2^{2/3}\cdot 2^{1/3}=2$$

13. $$\log_y x+\frac{1}{\log_y x}=2$$

$\Rightarrow$ $$\log_y x=1 \Rightarrow x=y$$

$$x^2+y=12$$

$\Rightarrow$ $$x^2+x-12=0$$

$\Rightarrow$ $$(x+4)(x-3)=0$$

$\Rightarrow$ $$x=-4 \text{ or } x=3$$

but $x>0$, then $x=3$

$$xy=9$$

14. $$y^x=x^y$$

if $x=2y$ then $y^{2y}=(2y)^y$

$\Rightarrow$ $$2y\log y=y\log(2y)$$

if $y\neq 0$ then $\log y^2=\log(2y)$

$\Rightarrow$ $$y^2=2y \Rightarrow y=2$$

$$x^2+y^2=5y^2=20$$

15. $(\log_2 4+\log_2(4^x+1))\log_2(4^x+1)=3$

Let $\log_2(4^x+1)=t$

$$t^2 + 2t - 3 = 0 \Rightarrow t = -3 \text{ or } 1$$

$$\log_2(4^x + 1) = 1 \Rightarrow 4^x = 1 \Rightarrow x = 0$$

17. $x^2 + 4x + 3 = 0 \quad (x > 0)$

18. $\log_{3^{1/4}}(\log_{3\sqrt{5}} x) = 4 \quad \Rightarrow \quad \log_3^{(\log_{3\sqrt{5}} x)} = 1$

$\Rightarrow \quad \log_5 x = 1 \quad \Rightarrow \quad x = 5$

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Chapter 17 – Straight Lines

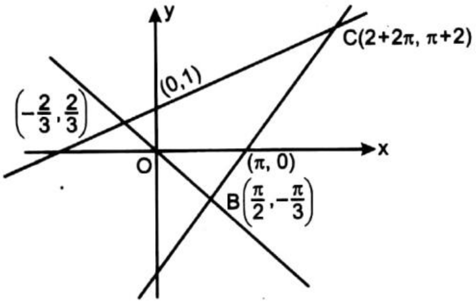
17

STRAIGHT LINES

Exercise-1 : Single Choice Problems

1. Let ratio be $\lambda : 1$ $\Rightarrow \dfrac{6\lambda - 3}{\lambda + 1} = 0$, $\lambda = \dfrac{1}{2}$

3.


y
x
C(2+2π, π+2)
(0,1)
$\left(-\frac{2}{3}, \frac{2}{3}\right)$
O
(π, 0)
B$\left(\frac{\pi}{2}, -\frac{\pi}{3}\right)$

if $(a, \sin a)$ lie inside the triangle, then $a \in (0, \pi)$

4.

$$x = \frac{711}{13 + 11m} = \frac{9 \times 79}{13 + 11m}$$

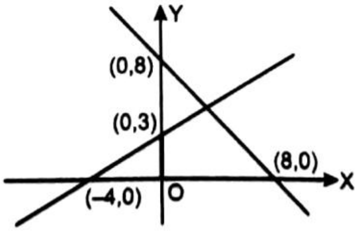
if x is an integer, then $m = 6$

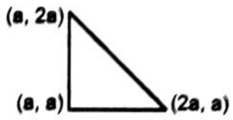
6.

$$7\left(\frac{y}{x}\right)^2 + 2c\left(\frac{y}{x}\right) - 1 = 0$$

$$m_1 + m_2 = 4m_1m_2 \Rightarrow c = 2$$

10.


Y
(0,8)
(0,3)
(8,0)
X
(–4,0)
O



13. $$\frac{1}{2}a^2 = 72$$

$$a = \pm 12$$

$$\text{Centroid} \equiv (16, 16) \text{ or } (-16, -16)$$

14. $$g(x) = ax + b$$

$$g(1) = 2$$

$\Rightarrow$ $$a + b = 2$$

$$g(3) = 0$$

$\Rightarrow$ $$2a = -2$$

$$a = -1$$

$$b = 3$$

$$g(x) = -x + 3$$

$$\cot[\cos^{-1}(|\sin x| + |\cos x|) - \sin^{-1}(|\sin x|) + |\cos x|]$$

$$|\sin x| + |\cos x| \in [1, \sqrt{2}]$$

$$\Rightarrow \cot[\cos^{-1} 1 - \sin^{-1} 1] = 0 = g(3)$$

15. Points A and B are mirror images about $y = x$.

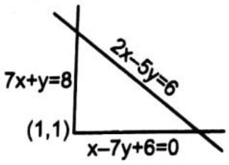
Point P will lie on the $\perp$ bisector of line joining A and B $\Rightarrow$ P lie on $y = x$.

16. $$4m^3 - 3am^2 - 8a^2m + 8 = 0 \begin{matrix} \nearrow m_1 \\ \rightarrow m_2 \\ \searrow m_3 \end{matrix}$$

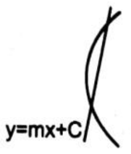
$$m_1 m_2 m_3 = -2$$

$\Rightarrow$ $$m_3 = 2 \qquad (\because m_1 m_2 = -1)$$

17.



18. $$2x^2 + 3y^2 - 5x\left(\frac{y - mx}{C}\right) = 0$$



Coefficient of x^2 + coefficient of $y^2 = 0$

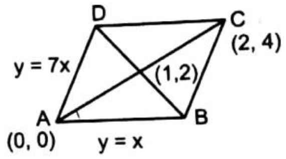
$$5 + \frac{5m}{C} = 0 \Rightarrow m + C = 0$$

Then the equation of family of line is $y = m(x - 1)$

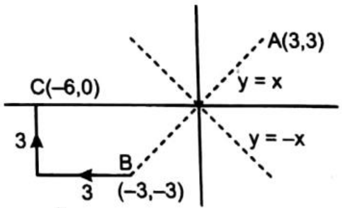
20. Equation of line BC is $y = 7x - 10$

Equation of line CD is $y = x + 2$

$$\text{Area of rhombus} = \left|\frac{(2-0)(10-0)}{(7-1)}\right| = \frac{10}{3}$$


D
C
(2, 4)
y = 7x
(1,2)
A
B
(0, 0)
y = x

21.

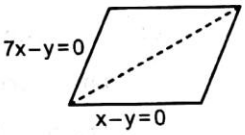

A(3,3)
y = x
C(–6,0)
y = –x
3
B
3
(–3,–3)

22. $y = \frac{3}{4}(x-9) + 6$

23. Acute angle bisector is

$$\frac{7x-y}{\sqrt{50}} = -\left(\frac{x-y}{\sqrt{2}}\right)$$

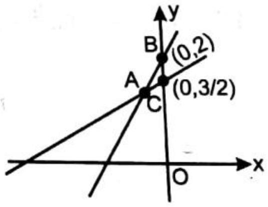
$\Rightarrow \qquad y = 2x$



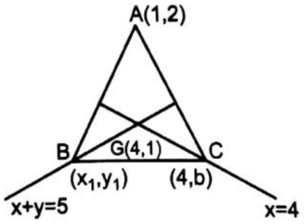
24. Either $x = y$ or $x = \left|\frac{3x+4y-12}{5}\right|$ or $y = \left|\frac{3x+4y-12}{5}\right| \Rightarrow (1, 1)$

25. Co-ordinate of point $A\left(-\frac{1}{7}, \frac{10}{7}\right)$

$$\text{Ar}(\Delta ABC) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{7} = \frac{1}{28}$$


y
B
(0,2)
A
(0,3/2)
C
O
X

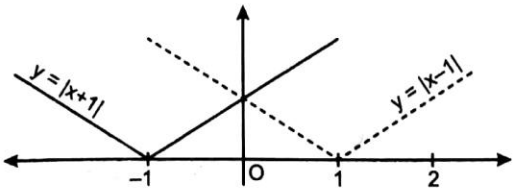
26.


A(1,2)
B
G(4,1)
C
(x1,y1)
(4,b)
x+y=5
x=4

Co-ordinate of centroid $G(4, 1) \Rightarrow \frac{x_1 + 4 + 1}{3} = 4$

$\Rightarrow \qquad x_1 = 7$ and $y_1 = -2$

27.



The image of $y = |x-1|$ w.r.t. y-axis is $y = |x+1| \Rightarrow y = \pm(x+1)$

Required solution $= (y-(x+1))(y+(x+1)) = 0$

28.

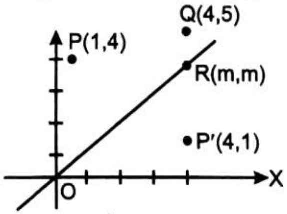
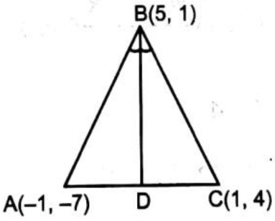


Image of $(1, 4)$ about the line $y = x$ is $(4, 1) \Rightarrow P'(4, 1)\, Q(4, 5)$ and $R(m, m)$ are collinear.

$\Rightarrow \qquad m = 4$

29. $\dfrac{AD}{CD} = \dfrac{AB}{BC} = \dfrac{10}{5} = \dfrac{2}{1}$



30. $4c\left(\dfrac{y}{x}\right)^2 - \left(\dfrac{y}{x}\right) + 6 = 0$ has one root is $-\dfrac{3}{4} \Rightarrow c = -3$

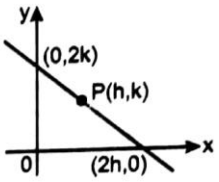
33.
$$\frac{x}{a} + \frac{y(a+c)}{2ac} + \frac{1}{c} = 0$$

$\Rightarrow \qquad a(y+2) + c(2x+y) = 0$

Passes through a fixed point $(1, -2)$

34.
$$\frac{1}{b}\left(\frac{y}{x}\right)^2 + \frac{2}{h}\left(\frac{y}{x}\right) + \frac{1}{a} = 0 \begin{array}{l} \nearrow m \\ \searrow 2m \end{array}$$

$\Rightarrow \qquad 3m = -\dfrac{2b}{h}$ and $2m^2 = \dfrac{b}{a} \Rightarrow \dfrac{ab}{h^2} = \dfrac{9}{8}$

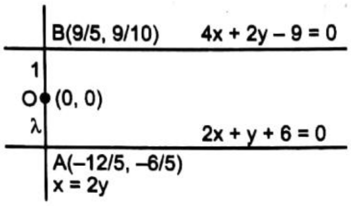

y
(0,2k)
P(h,k)
0
(2h,0)
x

35. Equation of line is $\frac{x}{2h}+\frac{y}{2k}=1$

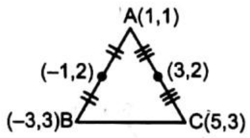
if it passes through fixed point (x_1, y_1)

$$\frac{x_1}{2h}+\frac{y_1}{2k}=1$$

36. $OA : OB = \lambda : 1 \Rightarrow \lambda = \frac{4}{3}$


B(9/5, 9/10)
4x + 2y – 9 = 0
1
O
(0, 0)
λ
2x + y + 6 = 0
A(–12/5, –6/5)
x = 2y

37. $G\left(1, \frac{7}{3}\right)$

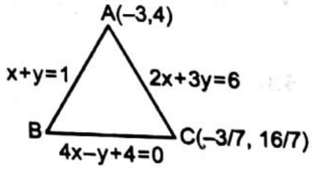

A(1,1)
(–1,2)
(3,2)
(–3,3)B
C(5,3)

38. Diagonals are perpendicular.

39. Let point on the line $x + y = 4$ is $(a, 4 - a)$.

$$\left|\frac{4(a)+3(4-a)-10}{5}\right|=1 \Rightarrow a^2+4a-21=0 \begin{matrix} \nearrow a_1 \\ \searrow a_2 \end{matrix}$$

$\Rightarrow \quad a_1 + a_2 = -4 \quad \Rightarrow b_1 + b_2 = 12$


A(–3,4)
x+y=1
2x+3y=6
B
C(–3/7, 16/7)
4x–y+4=0

40. Equation of altitude on BC

$$x + 4y = 13$$

Equation of altitude on AB

$$7x - 7y + 19 = 0$$

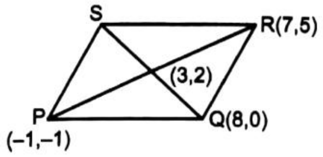
$\Rightarrow \qquad H\left(\frac{3}{7}, \frac{22}{7}\right)$

41. Equation of line is $(3x + 4y + 5) + \lambda(4x + 6y - 6) = 0$

$\Rightarrow \qquad \frac{-(3+4\lambda)}{4+6\lambda}\times\frac{7}{5}=-1 \quad \Rightarrow \quad \lambda = \frac{1}{2}$

42. $\frac{5-1}{8-2}=\frac{7-5}{x-8} \Rightarrow x = 11$

43.



$\Rightarrow \quad S(-2,4)$

44. $\text{Area} = \frac{1}{2}\begin{vmatrix} a & a & 1 \\ a+1 & a+1 & 1 \\ a+2 & a & 1 \end{vmatrix} = 1$

45. $(x-y)^2 = 1$

$\Rightarrow \quad x-y=1 \quad \text{and} \quad x-y+1=0$

46. *AB* subtend an acute angle at point *C*, then

$$a^2+(a+1)^2>4$$

$$\Rightarrow \quad a \in \left(-\infty, \frac{-\sqrt{7}-1}{2}\right) \cup \left(\frac{\sqrt{7}-1}{2}, \infty\right)$$

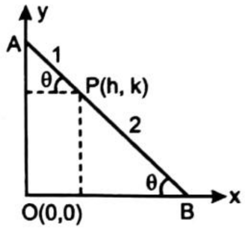
48.

$$h = \cos\theta$$

$$k = 2\sin\theta$$

$$h^2 + \frac{k^2}{4} = 1$$

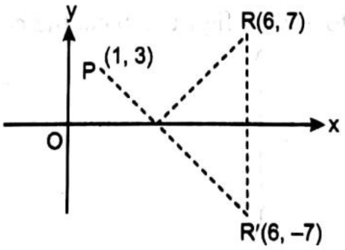
$$\Rightarrow \quad 4x^2 + y^2 = 4$$



50. Let the point of reflection is (h,k).

$$\frac{h-a}{1} = \frac{k-0}{-t} = \frac{-2(a+at^2)}{1+t^2} \Rightarrow x = -a$$

51.



52. Let (x,y) and (X,Y) be the old and the new coordinates, respectively. Since the axes are rotated in the anticlockwise direction, $\theta = +60°$. Therefore,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \dfrac{X}{2} - \dfrac{\sqrt{3}}{2}Y \\ \dfrac{\sqrt{3}}{2}X + \dfrac{Y}{2} \end{bmatrix}$$

$$\Rightarrow \qquad x = \frac{X}{2} - \frac{\sqrt{3}}{2}Y \text{ and } y = \frac{\sqrt{3}}{2}X + \frac{Y}{2}$$

$$\Rightarrow \quad \left(\frac{X}{2} - \frac{\sqrt{3}}{2}Y\right)^2 - \left(\frac{\sqrt{3}}{2}X + \frac{Y}{2}\right)^2 = a^2$$

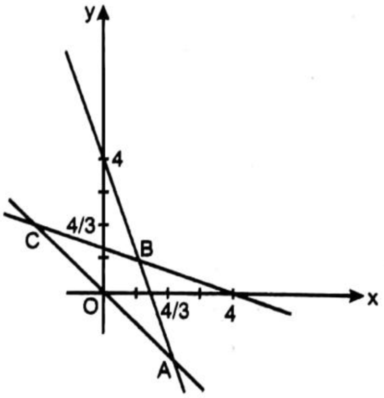
$$\Rightarrow \qquad (X^2 + 3Y^2 - 2\sqrt{3}XY) - (3X^2 + Y^2 + 2\sqrt{3}XY) = 4a^2$$

$$\Rightarrow \qquad -2X^2 + 2Y^2 - 4\sqrt{3}XY = 4a^2$$

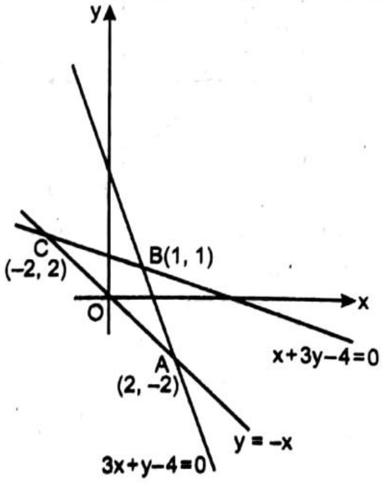
$$\Rightarrow \qquad Y^2 - X^2 - 2\sqrt{3}XY = 2a^2$$

which is the required equation.

53. The following figure depicts the condition. By observation from the figure, ΔABC is clearly an obtuse angled and isosceles triangle.



Alternate solution : The following figure depicts the condition.



From the figure, we get

$A: 3x + y = 4$ and $y = -x \Rightarrow x = 2;\ y = -2$

$B: (1, 1)$ by solving the equations.

$C: x + 3y - 4 = 0$ and $y = -x \Rightarrow x = -2;\ y = 2$

Thus,
$$AB = BC = \sqrt{1+9} = \sqrt{10}$$
$$AC = \sqrt{4^2 + 4^2} = 4\sqrt{2}$$
$$\cos B = \frac{10 + 10 - 16(2)}{2(\sqrt{10})(\sqrt{10})} < 0$$

Therefore, the given triangle is isosceles and obtuse angled triangle.

56. $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2} \Rightarrow$ Points are collinear.

57. $3h = a\cos t + b\sin t + 1$

$3k = a\sin t - b\cos t$

$\Rightarrow\ (3h - 1)^2 + (3k)^2 = (a\cos t + b\sin t)^2 + (a\sin t - b\cos t)^2 = a^2 + b^2$

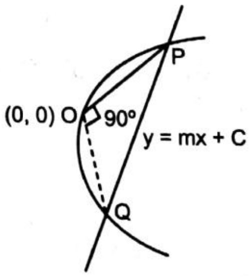
58. Equation of line $\frac{x}{a} + \frac{y}{-1-a} = 1.$

Lines passes from $(4, 3)$.

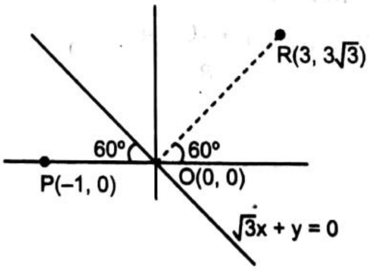
62. The given triangle is equilateral. Therefore, the orthocentre of the triangle is same as centroid of the triangle. Thus, the orthocentre, that is, the centroid is given by
$$\left(\frac{5 + 0 + (5/2)}{3}, \frac{0 + 0 + (5\sqrt{3}/2)}{3}\right) \equiv \left(\frac{5}{2}, \frac{5}{2\sqrt{3}}\right)$$

63. Using homogenization,
$$3x^2 - y^2 - 2x\left(\frac{y - mx}{C}\right) + 4y\left(\frac{y - mx}{C}\right) = 0$$

Coefficient of x^2 + Coefficient of $y^2 = 0$
$$\left(3 + \frac{2m}{C}\right) + \left(-1 + \frac{4}{C}\right) = 0$$
$$C = -m - 2$$



64.



Exercise-2 : One or More than One Answer Is/are Correct

1. Let line be $\frac{x}{a}+\frac{y}{b}=1$

$$a+b=9 \quad \text{and} \quad ab=20$$

$$\Rightarrow \quad a=5,\ b=4 \quad \text{or} \quad a=4,\ b=5$$

2. Centroid is $\left(4, \frac{4}{3}\right)$.

3. $\begin{vmatrix} 2 & 3 & -5 \\ t^2 & t & -6 \\ 3 & -2 & -1 \end{vmatrix} = 0 \Rightarrow t^2+t-6=0$

4.
$$b\left(\frac{y}{x}\right)^2+6\left(\frac{y}{x}\right)+a=0$$

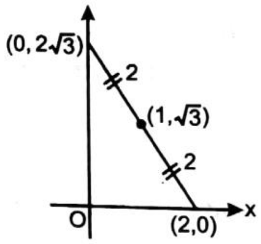
$$bm^2+6m+a=0$$

if $m=1$ is root of the equation

$$\Rightarrow \quad a+b=-6$$

if $m=-1$ is root of the equation

$$\Rightarrow \quad a+b=6$$

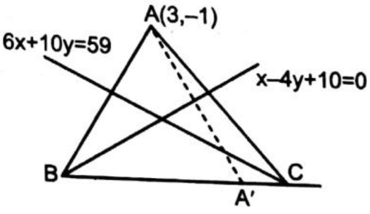


6. Co-ordinate of other two points

$$(1 \pm 2\cos\theta, \sqrt{3} \pm 2\sin\theta)$$

$$\left(1 \pm 2\left(\frac{\sqrt{3}}{2}\right), \sqrt{3} \pm 2\left(\frac{1}{2}\right)\right)$$

$$(1+\sqrt{3}, \sqrt{3}+1) \text{ and } (1-\sqrt{3}, \sqrt{3}-1)$$



8. Image of $A(3,-1)$ about angle bisector $x-4y+10=0$ is $A'(a,b)$.

$$\frac{a-3}{1}=\frac{b+1}{-4}=\frac{-2(3+4+10)}{17}$$

$$\Rightarrow \quad A'(1,7)$$

Let point $B\left(x_1, \frac{x_1+10}{4}\right)$ on the line $x-4y+10=0$

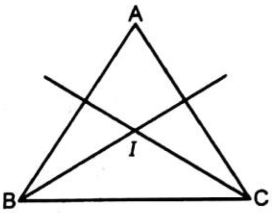
If mid-point of AB lie on the line $6x+10y=59$

$$6\left(\frac{x_1+3}{2}\right)+10\left(\frac{x_1+10-4}{8}\right)=59$$

$$\Rightarrow \quad B(10,5)$$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 3 to 4



3. $x + y = 2$ and $x - 3y = 6$

Meet at $(3, -1)$

4. Image of $A(2, -4)$ about $x + y = 2$ lie on BC.

$$\frac{x_2 - 2}{1} = \frac{y_2 + 4}{1} = -2\left(\frac{-4}{2}\right)$$

$$\Rightarrow \qquad x_2 = 6, y_2 = 0$$

Image of $A(2, -4)$ about $x - 3y = 6$ lie on BC.

$$\frac{x_3 - 2}{1} = \frac{y_3 + 4}{-3} = -2\frac{8}{10}$$

$$\Rightarrow \qquad x_3 = \frac{2}{5}, y_3 = \frac{4}{5}$$

Equ. of line BC, $\quad x + 7y = 6$

$$\Rightarrow \qquad B\left(\frac{4}{3}, \frac{2}{3}\right) \text{ and } C(6, 0)$$

Exercise-4 : Matching Type Problems

2. (A) $\sum_{r=1}^{n+1} ({}^1C_{r-1} + {}^2C_{r-1} + {}^3C_{r-1} + \ldots + {}^nC_{r-1})$

$$= \sum_{r=1}^{n+1} {}^1C_{r-1} + \sum_{r=1}^{n+1} {}^2C_{r-1} + \sum_{r=1}^{n+1} {}^3C_{r-1} + \ldots + \sum_{r=1}^{n+1} {}^nC_{r-1}$$

$$= 2^1 + 2^2 + 2^3 + \ldots + 2^n = 2(2^n - 1)$$

(B) Family of line $(x + y + 2) + \lambda(2x - y + 4) = 0$ always passes from $(-2, 0)$.

If almost one tangent can be drawn from $(-2, 0)$ then

$$S_1 = 4 - 8g - 36 + 4g^2 \le 0$$

$$g^2 - 2g - 8 \le 0$$

(C) $2\sin 7x \cdot \cos 2x = \cos 2x$

$$\Rightarrow \qquad \cos 2x = 0 \quad \text{or} \quad \sin 7x = \frac{1}{2}$$

$$x = \frac{\pi}{4}, \frac{3\pi}{4} \qquad 7x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}, \frac{37\pi}{6}, \frac{41\pi}{6}$$

(D) $a + b = \tan 65^\circ \tan 70^\circ - \tan 65^\circ - \tan 70^\circ$

$$\tan 135^\circ = \frac{\tan 65^\circ + \tan 70^\circ}{1 - \tan 65^\circ + \tan 70^\circ} = -1$$

$$\Rightarrow \quad \tan 65^\circ \tan 70^\circ - \tan 65^\circ - \tan 70^\circ = 1$$

3. (A) $\cos 40^\circ - 2\cos 40^\circ \sin 10^\circ = \cos 40^\circ - (\sin 50^\circ - \sin 30^\circ)$

(B) $\begin{vmatrix} 1 & 1 & 1 \\ 3 & 2\lambda & 4 \\ 1 & 1 & -3\lambda \end{vmatrix} = 0 \quad \Rightarrow (3 - 2\lambda)(1 + 3\lambda) = 0$

(C) $\begin{vmatrix} k & 2-2k & 1 \\ -k+1 & 2k & 1 \\ -4-k & 6-2k & 1 \end{vmatrix} = 0 \quad \Rightarrow 2k^2 + k - 1 = 0$

$$\Rightarrow k = -1, \frac{1}{2}$$

(D) $\sum_{k=3}^{\infty}\left(\frac{1}{2}\right)^k = \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5 + \ldots\ldots = \frac{1}{4}$

Exercise-5 : Subjective Type Problems

1. $\Delta = 132$

2.
$$ax + by + c = 3x - 4y + c$$

$\Rightarrow \quad a = 3, b = -4$

Distance of $3x - 4y + c$ from $A(3, 1)$ is 1.

$$\Rightarrow \quad \frac{|9 - 4 + c|}{5} = 1$$

$$|c + 5| = 5$$

Also, $3x - 4y + c = 0$ and $3x - 4y + 5 = 0$ lie on same side of A

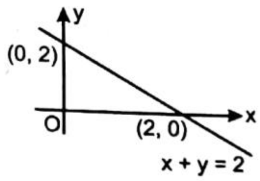
$\Rightarrow \quad c + 5 > 0$

$\Rightarrow \quad c + 5 = 5 \;\Rightarrow\; c = 0$

3. $xy(x + y - 2) = 0$

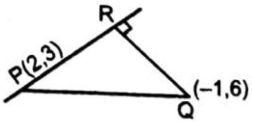
$\alpha + \alpha^4 - 2 \le 0 \qquad (\alpha > 0)$

$\Rightarrow \quad \alpha = 1$



5. $PQ = 3\sqrt{2}$

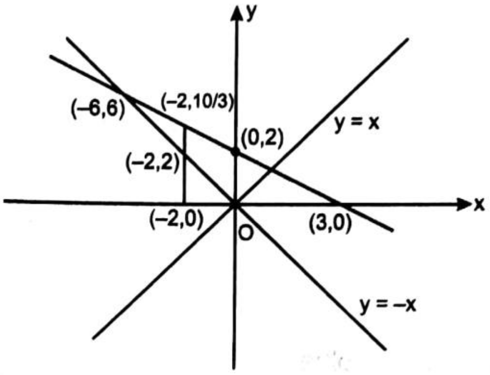
$QR \le PQ$



6. $x^2(y^2 - x^2) = 0$

has 3 different lines $x = 0, y = x$ and $y = -x$.

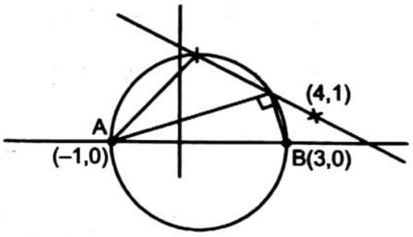
8. $2 < a < \dfrac{10}{3}$


y
x
(–6,6)
(–2,10/3)
(0,2)
y = x
(–2,2)
(–2,0)
O
(3,0)
y = –x

9. Describe a circle whose diameter is AB.

$\therefore$ centre $= (1, 0)$

Radius $= 2$



Let 'm' the slope of the line passing through $(4, 1)$.

$(y - 1) = m(x - 4)$ intersect the circle

$\perp$ distance from centre < radius of circle.

$$\left|\frac{-3m+1}{\sqrt{m^2+1}}\right| < 2$$

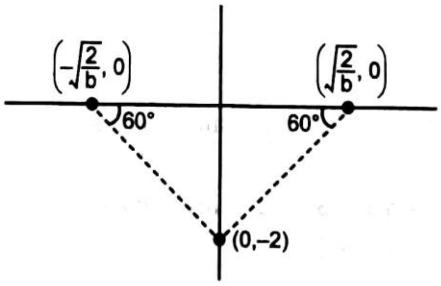
$$9m^2 - 6m + 1 < 4m^2 + 4$$

$$\Rightarrow \qquad m \in \left(\frac{6-\sqrt{96}}{10}, \frac{6+\sqrt{96}}{10}\right) - \left\{\frac{1}{5}, 1\right\}$$

$$\lambda_1 + \lambda_2 = \frac{12}{10} = \frac{6}{5}$$

$$5(\lambda_1 + \lambda_2) = 6$$

10. $\sqrt{\dfrac{2}{b}} = \dfrac{2}{\sqrt{3}} \Rightarrow b = \dfrac{3}{2}$


$\left(-\sqrt{\frac{2}{b}}, 0\right)$
$\left(\sqrt{\frac{2}{b}}, 0\right)$
60°
60°
(0,–2)

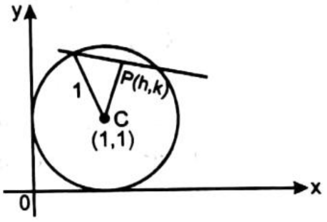
* * *

Chapter 18 – Circle

18

CIRCLE

Exercise-1 : Single Choice Problems

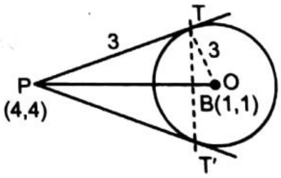


1. $CP = \frac{\sqrt{3}}{2} = \sqrt{(h-1)^2 + (k-1)^2}$

Locus of point $P(h, k)$ is

$$(x-1)^2 + (y-1)^2 = \frac{3}{4}$$

2. $\sqrt{d^2 - (r_1 - r_2)^2} = 15; \quad \sqrt{d^2 - (r_1 + r_2)^2} = 5 \Rightarrow r_1 r_2 = 50$

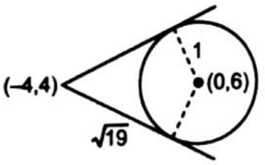


4. $$PT = \sqrt{16 + 16 - 8 - 8 - 7} = 3$$

$\Rightarrow$ $TT' = 2BT = 2 \cdot 3 \cos 45° = 3\sqrt{2}$

5. It will be circle with diametric ends as $(1, 1)$ and $(4, 2)$ *i.e.*, point of intersection.

6.



8. Let centroid be (h, k).

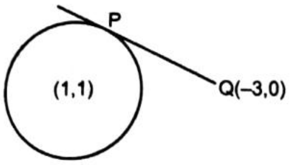
$\Rightarrow$ $$h = \frac{\cos\alpha + \sin\alpha + 1}{3}, \quad k = \frac{\sin\alpha - \cos\alpha + 2}{3}$$

$\Rightarrow$ $3h - 1 = \cos\alpha + \sin\alpha, \quad 3k - 2 = \sin\alpha - \cos\alpha$

$\Rightarrow$ $(3h-1)^2 + (3k-2)^2 = 2$

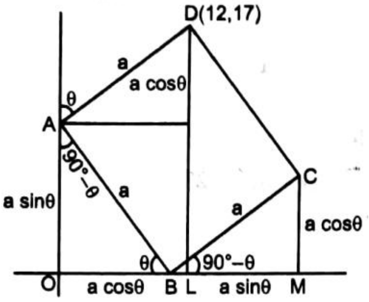
$\Rightarrow$ $$\left(x - \frac{1}{3}\right)^2 + \left(y - \frac{2}{3}\right)^2 = \frac{2}{9}$$

9.



Length of tangent $= PQ = \sqrt{4^2 + 1^2 - 5} = \sqrt{12}$

10.

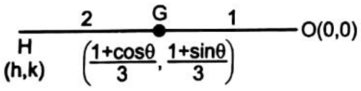

D(12,17)
a
a cosθ
θ
A
90°–θ
a
a sinθ
C
a
a cosθ
θ
90°–θ
O
a cosθ
B
L
a sinθ
M

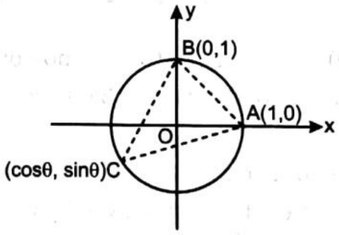
$OA = a\sin\theta = 12, \quad DL = a\sin\theta + a\cos\theta = 17$

$$a\cos\theta = 5$$

$$C = (a\cos\theta + a\sin\theta, a\cos\theta) = (17, 5)$$

12. Centroid divide the line joining orthocentre and circumcentre in 2 : 1.


2
G
1
O(0,0)
H
(h,k)
$\left(\frac{1+\cos\theta}{3}, \frac{1+\sin\theta}{3}\right)$


y
B(0,1)
A(1,0)
x
O
(cosθ, sinθ)C

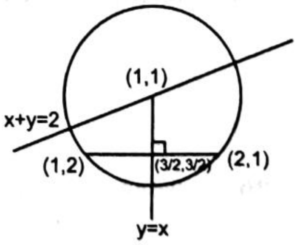
$\Rightarrow \qquad h = 1 + \cos\theta, \quad k = 1 + \sin\theta$

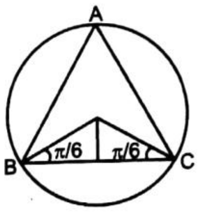
$$(x-1)^2 + (y-1)^2 = 1$$

13. Co-ordinate of centre is $C(1, 1)$.

$$(x-1)^2 + (y-1)^2 = 1$$

$$x^2 + y^2 - 2x - 2y + 1 = 0$$


(1,1)
x+y=2
(1,2)
(3/2,3/2)
(2,1)
y=x

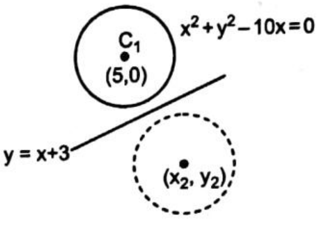


14.

$$a = 2R\cos\frac{\pi}{6}$$

$$\Rightarrow \qquad a = 4\sqrt{3}\text{ cm}$$

$$\text{Area of } \Delta ABC = \frac{\sqrt{3}}{4}a^2 = 12\sqrt{3}\text{ cm}^2$$



15. Image of centre $C_1(5,0)$ about the line $y = x + 3$ is

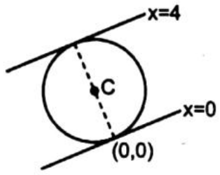
$$\frac{x_2 - 5}{1} = \frac{y_2 - 0}{-1} = \frac{-2(5+3)}{1^2 + 1^2}$$

$$\Rightarrow \qquad C_2(-3,8)$$

Equation of reflected circle is

$$(x+3)^2 + (y-8)^2 = 25$$

19.



20. Let the equation of line is $3x + 4y = C$

$$\left|\frac{C}{5}\right| = 3 \quad \Rightarrow C = 15 \text{ (in first quadrant)}$$

21. $C_1(5,0), C_2(3,-1), C_3(3/2, 2)$ do not lie on straight line.

22. Let equation of diameter is $3x + 5y = C$

$$\Rightarrow \qquad C = 7$$

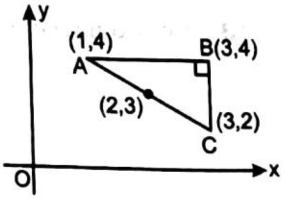
23. Equation of circle is

$$(x+1)(x-2) + (y-2)(y-3) + \lambda(x - 3y + 7) = 0$$

If its radius is $\sqrt{5}$.

$$\Rightarrow \qquad \lambda = \pm 1$$

25. Equation of circle is $(x-1)(x-3) + (y-4)(y-2) = 0$

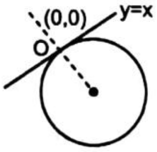


26. Equation of tangent at $O(0,0)$.

$$x(0) + y(0) + g(x+0) + f(y+0) = 0$$

$$\Rightarrow \qquad gx + fy = 0$$

27. Equation of normal at $O(0,0)$



$$y=-x$$

$$\text{Centre}\left(0\pm\left(-\frac{1}{\sqrt{2}}\right),0\pm\left(\frac{1}{\sqrt{2}}\right)\right)$$

$$\Rightarrow \qquad \text{Either}\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\text{or}\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$$

28. Here, $C_1C_2=r_1+r_2$ (Condition for external touch)

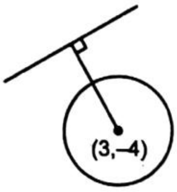
30. The triangle is right angled and the radical centre will be the orthocentre of the triangle.

32. Equation of common chord is $6x+14y+(l+m)=0$

If it passes through $(1,-4)$. Then, $l+m=50$

33.

$$x^2+y^2-6x+8y=0$$

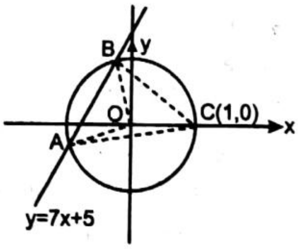


Distance of line from centre

$$\left|\frac{9-16-25}{5}\right|=\frac{32}{5}$$

$$\text{Shortest distance}=\frac{32}{5}-5=\frac{7}{5}$$

34. $\angle AOB=\frac{\pi}{2} \Rightarrow \angle ACB=\frac{\pi}{4}$



37. Equation of required circle :

$S:(x-1)^2+(y-1)^2+\lambda(x-y)=0$

$S':x^2+y^2+2y-3=0$

Common chord of $S=0$ and $S'=0$ is $S-S'=0$

$$(\lambda-2)x-(\lambda+4)y+5=0$$

Centre of $S':(0,-1)$ lies on common chord $\Rightarrow \lambda=-9$

$S:(x-1)^2+(y-1)^2-9(x-y)=0$

$$\Rightarrow \qquad r=\frac{9}{\sqrt{2}}$$

40. Point lie inside the circle $k^2+(k+2)^2<4 \Rightarrow 2k^2+4k<0; -2<k<0$

41. The length of the normal is

$$\left|y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right|$$

The length of radius vector of a point (x, y) on the curve is $|xi + yi|$, that is $\sqrt{x^2+y^2}$, it is given that

$$\sqrt{x^2+y^2} = |y|\sqrt{1+(y')^2}$$

Squaring on both sides of this equation, we get

$$x^2+y^2 = y^2[1+(y')^2]$$

$$\Rightarrow \quad x^2+y^2 = y^2 + y^2\left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \quad x^2 = \left(y\frac{dy}{dx}\right)^2$$

$$\Rightarrow \quad y\frac{dy}{dx} = x \text{ or } y = \frac{dy}{dx} = -x$$

Now, $$y\frac{dy}{dx} = x$$

$$\Rightarrow \quad y\,dy = x\,dx$$

Integrating on both sides, we get

$$\frac{y^2}{2} = \frac{x^2}{2} + c$$

$$\Rightarrow \quad x^2 - y^2 = 2c \text{ or } x^2 - y^2 = \text{constant}$$

This answer does not exist in the given options. So, consider the other alternative.

$$y\,dy = -x\,dx$$

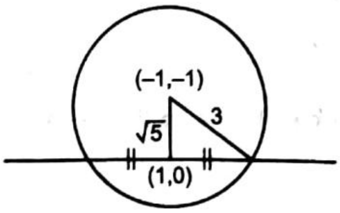
Integrating on both sides, we get

$$\frac{y^2}{2} = -\frac{x^2}{2} + c$$

$$\Rightarrow \quad x^2 + y^2 = \text{constant}$$

and this constant is > 0 in practical sense.

44. Length of chord $= 2\sqrt{3^2-5} = 4$



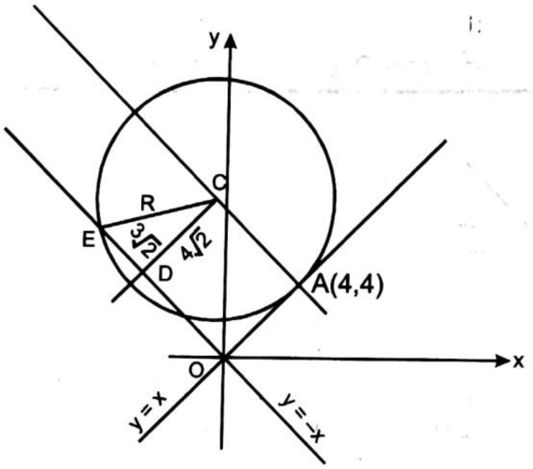
47. Family of circles touching the line $y = x$ at the point $(4, 4)$ is

$$(x-4)^2 + (y-4)^2 + \lambda(y-x) = 0$$

We need to find the member of this family which has length of chord $= 6\sqrt{2}$ on $x + y = 0$. For different λ's, we get different circles.

$$x^2 + y^2 - 8x - 8y + 32 + \lambda y - \lambda x = 0$$

$$\Rightarrow \quad x^2 + y^2 + x(-8-\lambda) + y(-8+\lambda) + 32 = 0 \qquad \ldots(1)$$



Now,

$$OA = DC = 4\sqrt{2}$$

$$DE = 3\sqrt{2} = \frac{6\sqrt{2}}{2} \text{ (given)}$$

Therefore, $\quad R^2 = (3\sqrt{2})^2 + (4\sqrt{2})^2$

$$\Rightarrow \quad \frac{\lambda^2}{2} = 50 \Rightarrow \lambda^2 = 100 \Rightarrow \lambda = \pm 10$$

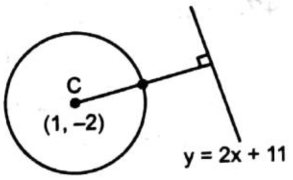
Substituting $\lambda = -10$ in eq. (1), we get

$$x^2 + y^2 + 2x - 18y + 32 = 0$$

[Substituting $\lambda = 10$; in eq. (1); we get $x^2 + y^2 - 18x + 2y + 32 = 0$, which does not exist in the given options]

Note : From eq. (1), we get

$$R^2 = (\text{Radius})^2 = g^2 + f^2 - c = \frac{(\lambda+8)^2}{4} + \frac{(\lambda-8)^2}{4} - 32 = \frac{\lambda^2}{2}$$



48. Slope of line normal to circle and perpendicular to line

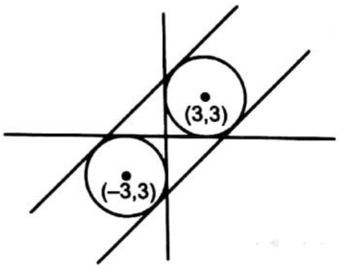
$$m=-\frac{1}{2}=\tan\theta$$

Co-ordinate of point lie on normal at a dist. of 3 from centre

$$\left(1\pm 3\left(\frac{-2}{\sqrt{5}}\right),-2\pm 3\left(\frac{1}{\sqrt{5}}\right)\right)$$

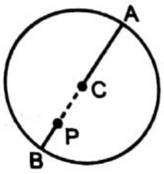
Exercise-2 : One or More than One Answer is/are Correct

2.



3. $x^2+y^2-x\left(\frac{\pi}{2}\right)+y\left(\frac{\pi}{2}-2\sin^{-1}\alpha\right)=0$

$\Rightarrow$ Length of chord $=2\sqrt{\left(\frac{\pi}{4}\right)^2+\left(\frac{\pi}{4}-\sin^{-1}\alpha\right)^2}$



7. $(x+2)^2+(y-3)^2$ is nothing but square of distance between (x,y) and $(-2,3)$ where (x,y) is point lies on the circle.

Centre $=(-4,5)$, $r=\sqrt{16+25+40}=9$

Clearly, $(-2,3)$ is lies inside the circle.

$\therefore$ $PC=2\sqrt{2}$ $a=PA^2=(9+2\sqrt{2})^2$

$b=PB^2=(9-2\sqrt{2})^2$

$\therefore$ $a+b=178,\ a-b=72\sqrt{2}$

8. Let point of intersection $P(h,k)$

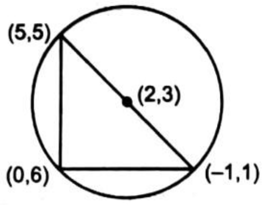
Equation of chord of contact is

$$hx+ky=a^2$$

If it is tangent to $x^2+y^2-2ax=0$

$$\Rightarrow \left|\frac{ha-a^2}{\sqrt{h^2+k^2}}\right|=a$$

9. Equation of tangent to circle



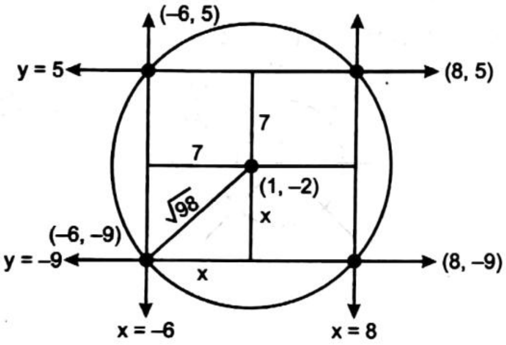
$$y-3=\frac{3}{2}(x-2)\pm\sqrt{13}\sqrt{1+\frac{9}{4}}$$

$$\Rightarrow\ 2y=3x+13,\ \ 2y=3x-13$$

$$\frac{x_2-2}{3}=\frac{y_2-3}{2}=-\left(\frac{13}{13}\right)\qquad\Rightarrow(-1,5)$$

$$\frac{x_3-2}{3}=\frac{y_3-3}{-2}=-\left(\frac{-13}{13}\right)\qquad\Rightarrow(5,1)$$

10. $2x^2=98\Rightarrow x^2=49\Rightarrow x=\pm 7$


(−6, 5)
y = 5
(8, 5)
7
7
(1, −2)
√98
x
(−6, −9)
y = −9
x
(8, −9)
x = −6
x = 8

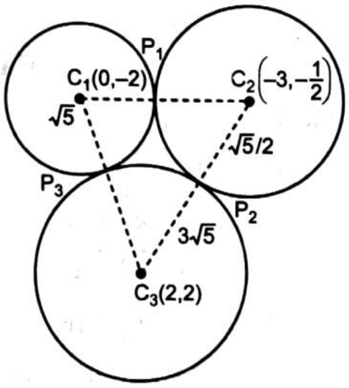
Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 3

Sol. $P_1(-2,-1)$

$P_2\left(-\frac{16}{7},-\frac{1}{7}\right)$

$P_3\left(\frac{1}{2},-1\right)$


P1
C1(0,−2)
C2(−3,−1/2)
√5
√5/2
P3
P2
3√5
C3(2,2)

Paragraph for Question Nos. 4 to 6

4. $S:x^2+y^2+x(2\lambda-9)+y(3\lambda-12)+53-27\lambda=0$

$C:x^2+y^2-4x-6y-3=0$

Equation of line : $S-C=0$

or $x(2\lambda-5)+y(3\lambda-6)+56-27\lambda=0$

or $5x+6y-56=0$ or $2x+3y-27=0$

$\Rightarrow$ $x=2,\quad y=\dfrac{23}{3}$

5. Centre of C lies on common chord of S and C.

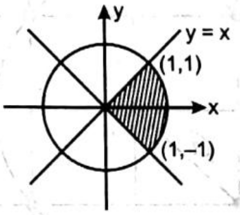
$\Rightarrow$ $(2,3)$ lies on $x(2\lambda-5)+y(3\lambda-6)+56-27\lambda=0$

$\Rightarrow$ $S: x^2+y^2-5x-6y-1=0$

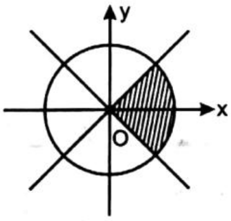
6. Difference of squares of lengths of tangents from A and B is 3, which is equal to $|AP^2-BP^2|$.

Paragraph for Question Nos. 7 to 8

7. Max. dist. between any two arbitrary points = 2



8.



Paragraph for Question Nos. 9 to 10

Sol. Let $P(h,k)$

$L_1=\sqrt{h^2+k^2-4}$

$L_2=\sqrt{h^2+k^2-4h}$

$L_3=\sqrt{h^2+k^2-4k}$

If $L_1^4=L_2^2L_3^2+16$

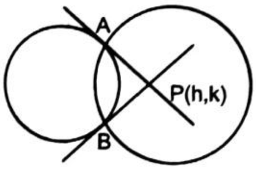
$\Rightarrow$ $(h^2+k^2-4)^2=(h^2+k^2-4h)(h^2+k^2-4k)+16$

$\Rightarrow$ $(h+k)(h^2+k^2-2h-2k)=0$

$C_1: x+y=0$

$C_2: x^2+y^2-2x-2y=0$

Exercise-5 : Subjective Type Problems



1. Equation of chord of contact w.r.t. P

$$hx + ky = 1$$

Equation of common chord is

$$(\lambda - 3)x + (2\lambda + 2)y + 3 = 0$$

$$\Rightarrow \quad \frac{\lambda - 3}{h} = \frac{2\lambda + 2}{k} = -3$$

$\Rightarrow$ Equation of locus is $6x - 3y - 8 = 0$

2. By using system of circles any circle passing through (1, 1) and (–2, 1) is

$$(x-1)(x+2) + (y-1)^2 + \lambda(y-1) = 0 \qquad \ldots(1)$$

Given circles $\quad x^2 + y^2 - 1 = 0 \qquad \ldots(2)$

Now radical axis of (1) and (2) is

$$(x - 2y) + \lambda(y-1) = 0 \qquad \ldots(3)$$

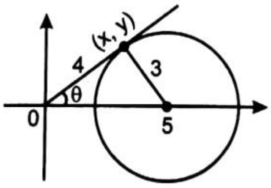
$\because$ Radical centre of given circles is $(0, 0)$.

So, eq. (3) is passing through $(0, 0)$.

$\therefore \quad \lambda = 0$

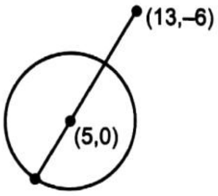
Put $\lambda = 0$ in eq. (1) we get required circle.

3. $\frac{y}{x} = \tan\theta = \frac{3}{4}$



4. $x^2 + y^2 - 26x + 12y + 210$

$(x-13)^2 + (y+6)^2 + 5$



5. $\quad S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$

$$\Rightarrow \quad 2g + 2f = -c - 2 \qquad \ldots(1)$$

(1, 1) satisfy circle.

$$\Rightarrow \quad 2g + 2f + c = -2$$

$$\Rightarrow \quad c = 0$$

and $$g+f=-1$$

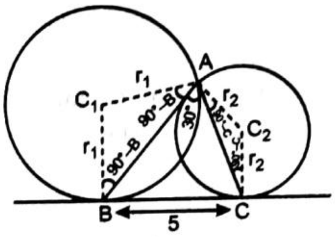
∴ Length of tangent $=\sqrt{8+4g+4f+c}=2$

6. Length of common external tangent

$$\sqrt{d^2-(r_1-r_2)^2}=5 \qquad \ldots(1)$$

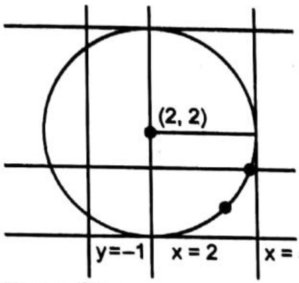
$$\cos(90^\circ-B+90^\circ-C+30^\circ)=\cos 60^\circ$$

$$=\frac{r_1^2+r_2^2-d^2}{2r_1r_2} \qquad \ldots(2)$$



A
C₁
C₂
r₁
r₁
r₂
r₂
90°-B
90°-B
30°
90°-C
90°-C
B
C
5

9.

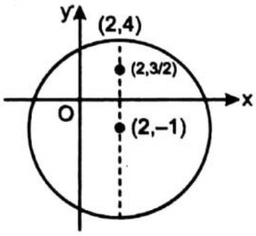


From diagram common points are 3.

10. $(C_1C_2)^2=r_1^2+r_2^2$

$$18=2r^2 \Rightarrow r^2=9$$

11.

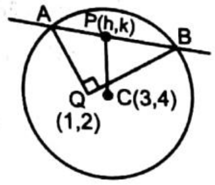


y
x
O
(2,4)
(2,3/2)
(2,−1)

12. $PQ=PA=PB$

$$\sqrt{(h-1)^2+(k-2)^2}=\sqrt{6^2-(h-3)^2-(k-4)^2}$$

$$\Rightarrow \quad h^2+k^2-4h-6k-3=0$$



A
P(h,k)
B
Q
(1,2)
C(3,4)

13. $c=3, a^2+b^2=36$

Length of chord $AB=2\sqrt{r^2-p^2}$

$$=2\sqrt{c-\left(\frac{2c}{\sqrt{a^2+b^2}}\right)^2}=2\sqrt{2}$$

□□□

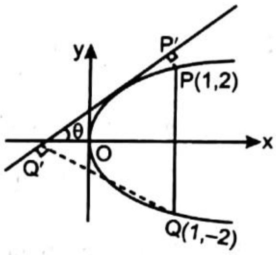
Chapter 19 – Parabola

19

PARABOLA

Exercise-1 : Single Choice Problems

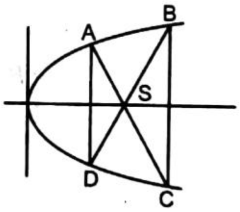
1.
$$P'Q' = PQ\cos(90^\circ - \theta)$$
$$= \frac{4}{\sqrt{t^2+1}} (t^2 < 1)$$
$$(P'Q')_{\min} = 2\sqrt{2}$$



2. Equation of circle with SP as diameter
$$(x-4)\left(x-\frac{9}{4}\right) + y(y-6) = 0$$
$$\text{Centre}\left(\frac{25}{8}, 3\right) \text{ and radius} = \frac{25}{8}$$

Equation of normal at $P(4, 6)$ is
$$4x + 3y = 34$$
$$\text{Length of chord} = 2\sqrt{\left(\frac{25}{8}\right)^2 - \left(\frac{\frac{25}{8} + 9 - 34}{5}\right)^2} = \frac{15}{4}$$

3. The diagonals are the focal chord.
$$AS = 1 + t^2 = c \text{ (say)}$$
$$\frac{1}{c} + \frac{1}{\left(\frac{25}{4} - c\right)} = 1 \qquad \left(\because \frac{1}{AS} + \frac{1}{CS} = \frac{1}{a}\right)$$
$$\Rightarrow \qquad c = \frac{5}{4}, 5$$



$A\left(\frac{1}{4}, 1\right)$, $B(4, 4)$, $C(4, -4)$ and $D\left(\frac{1}{4}, -1\right)$

Area of trapzium $= \frac{1}{2}(2 + 8) \times \frac{15}{4}$

4. For normal chord $t_2 = -t_1 - \frac{2}{t_1}$

Also chord substends an angle of 90° at the vertex

$\therefore \quad t_1 t_2 = -4 \quad \Rightarrow \quad t_2^2 = 8$

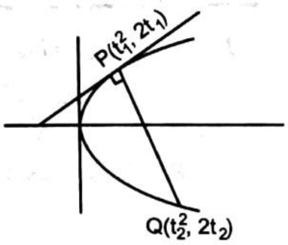
9. $(y - x + 2) + \lambda(y + x - 2) = 0$

The family of lines passes through $(2, 0)$.

The chord is $x = 2$ and end points are $(2, \pm 4)$.

10.
$$t_2 = -t_1 - \frac{2}{t_1}$$

$$h = \frac{t_1^2 + t_2^2}{2} \text{ and } k = \frac{2t_1 + 2t_2}{2}$$



Put the value of t_2 and eliminate t_1 we get

$$h - 2 = \frac{4}{k^2} + \frac{k^2}{2} \Rightarrow a = 2, b = 4, c = 2$$

11. The parabola is $(y - 1)^2 = 4(x - 1)$. The coordinates of $P(1 + t_1^2, 1 + 2t_1)$ and $Q(1 + t_2^2, 1 + 2t_2)$.

Here $S(2, 1)$ is the focus. The coordinates of T are G.M. of abscissa and A.M. of ordinates of P and Q.

$\Rightarrow \quad ST^2 = 16 \qquad \therefore \; SP \cdot SQ = ST^2$

12. Let $P(t_1)$ and $Q(t_2)$ are point of $y^2 = 8x$

$$2t_1^2 + 2t_2^2 = 17 \text{ and } (2t_1^2)(2t_2^2) = 11$$

$$ST^2 = SP \cdot SQ = 2(1 + t_1^2)\, 2(1 + t_2^2) = 34 + 4 + 11$$

$$ST = \sqrt{49}$$

13. $ay = x^2 \Rightarrow \dfrac{-1}{\left(\dfrac{dy}{dx}\right)} = \dfrac{-a}{2x_1} = -\dfrac{A}{B}$ (slope of normal)

$\Rightarrow x_1 = \frac{aB}{2A}$ and $y_1 = \frac{1}{B} - \frac{a}{2}$ put (x_1, y_1) in $ay = x^2$

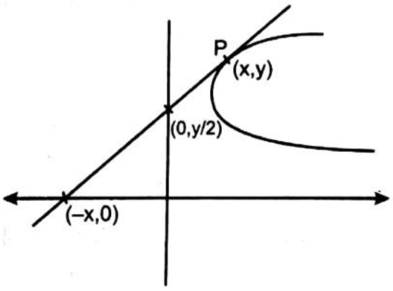
14.

$$\frac{dy}{dx} = \frac{y}{2x}$$

$$\frac{2dy}{y} = \frac{1}{x}dx$$

$$\Rightarrow \quad 2\log y = \log x + \log c$$

$$\Rightarrow \quad y^2 = cx \quad \text{put } (3,1)$$



15.

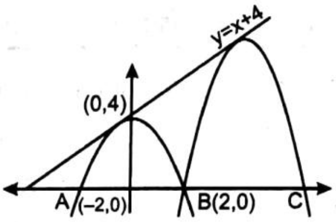
$$(x-\alpha)^2 = -(y-(\alpha+4))$$

The curve passes through $(2,0)$

$$(2-\alpha)^2 = -(0-(\alpha+4))$$

$$\alpha^2 - 5\alpha = 0 \Rightarrow \alpha = 0 \text{ or } \alpha = 5$$

$$(x-5)^2 = -(y-9) \quad \text{put } y=0 \Rightarrow x = 2, 8$$

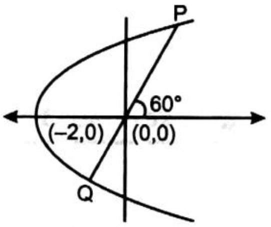


16. $y = (\tan 60°)x$ is the focal chord.

Coordinates of P and Q are intersection of $y = \sqrt{3}x$ with parabola

$$P(4, 4\sqrt{3}),\ Q\left(-\frac{4}{3}, \frac{-4}{\sqrt{3}}\right)$$

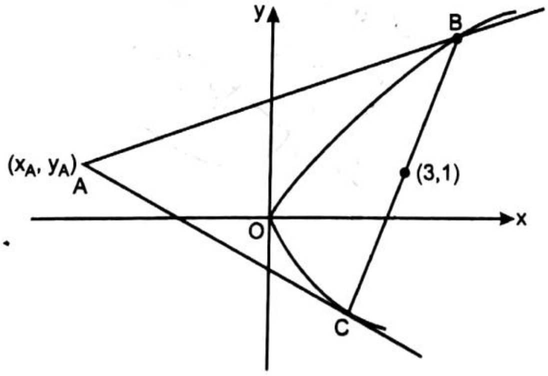
Find ⊥ bisector of PQ.



17. The director circle of the parabola is its directrix $(x+11=0)$. Now apply condition of tangency.

18. The following figure depicts the condition. Chord of contact of a point $A(x_A, y_A)$ with respect to $y^2 = 4x$ is $y_A y = 2(x + x_A)$. Since this chord passes through the point (3, 1), we have

$$y_A = 2(x_A + 3)$$



AB and AC are tangents to the parabola.

BC is chord of contact of point A with respect to the parabola $y^2 = 4ax$.

Given that point A lies on $x^2 + y^2 = 25$, we have

$$x_A^2 + y_A^2 = 25$$

$$\Rightarrow \quad x_A^2 + 4(x_A + 3)^2 = 25$$

$$\Rightarrow \quad x_A^2 + 4(x_A^2 + 9 + 6x_A) = 25$$

$$\Rightarrow \quad 5x_A^2 + 24x_A + 36 - 25 = 0$$

$$\Rightarrow \quad 5x_A^2 + 24x_A + 11 = 0$$

Exercise-2 : One or More than One Answer is/are Correct

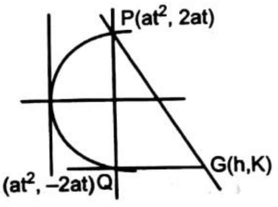
1. Equation of normal at $P(at^2, 2at)$ is

$$y = -tx + 2at + at^3$$

$$G(4a + at^2, -2at)$$

$\Rightarrow$ Locus of point $G(h, K)$ is

$$y^2 = 4a(x - 4a)$$



Exercise-3 : Comprehension Type Problems

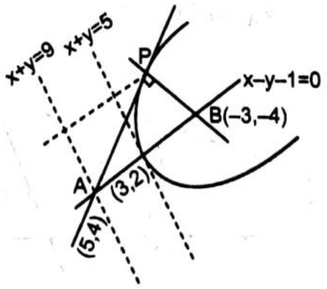
Paragraph for Question Nos. 1 to 3

Sol. Tangent and normal are angle bisectors of focal radius and perpendicular to directrix.

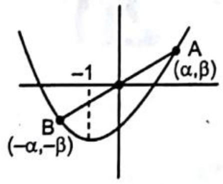
Circle 'C' circumscribing ΔABP is

$$(x - 5)(x + 3) + (y - 4)(y + 4) = 0$$

Length of latus rectum $= 4(2\sqrt{2}) = 8\sqrt{2}$



Exercise-5 : Subjective Type Problems



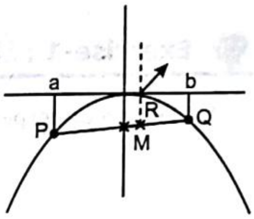
1.
$$\beta = 2\alpha^2 + 4\alpha - 2 \qquad \ldots(1)$$
$$-\beta = 2\alpha^2 - 4\alpha - 2 \qquad \ldots(2)$$

(1) & (2) $\Rightarrow 4\alpha^2 - 4 = 0 \Rightarrow \alpha = \pm 1$

Put $\alpha = 1, \beta = 2 + 4 - 2 = 4$

$\therefore \quad A(1, 4), B(-1, -4)$

$$AB^2 = l^2 = (\sqrt{4+64})^2 = 68$$



2. $R = \left(\dfrac{a+b}{2}, -\left(\dfrac{a+b}{2}\right)^2\right)$, $M = \left(\dfrac{a+b}{2}, \dfrac{-a^2-b^2}{2}\right)$

$$PQ = y + b^2 = \frac{-b^2 + a^2}{b-a}(x-b)$$
$$y + b^2 = -(b+a)(x-b)$$
$$y = -(b+a)(x-b) - b^2$$
$$\Delta_1 = \int_a^b [[-(a+b)(x-b) - b^2] + x^2]\,dx$$
$$= -(a+b)\frac{(x-b)^2}{2} - b^2x + \frac{x^3}{3}\Bigg|_a^b = \frac{(a-b)^3}{6}$$

Area of $\Delta PQR = \Delta_2 = \dfrac{1}{2}\begin{vmatrix} a & -a^2 & 1 \\ b & -b^2 & 1 \\ \dfrac{a+b}{2} & -\left(\dfrac{a+b}{2}\right)^2 & 1 \end{vmatrix}$

$R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$, we get $\Delta_2 = \dfrac{(a-b)^3}{8}$

3.
$$m_{AB} \times m_{BC} = -1$$

$\Rightarrow \quad \dfrac{-2}{(t_1+t_2)} \times \dfrac{-2}{(t_2+t_3)} = -1$

$\Rightarrow \quad (t_1+t_2)(t_2+t_3) = -4$

Similarly,
$$m_{AD} \times m_{CD} = -1$$

$\Rightarrow \quad (t_1+t_4)(t_3+t_4) = -4$

$\Rightarrow \quad (t_1+t_2)(t_2+t_3) = (t_1+t_4)(t_3+t_4)$

Solving this
$$\frac{t_2+t_4}{t_1+t_3} = -1$$

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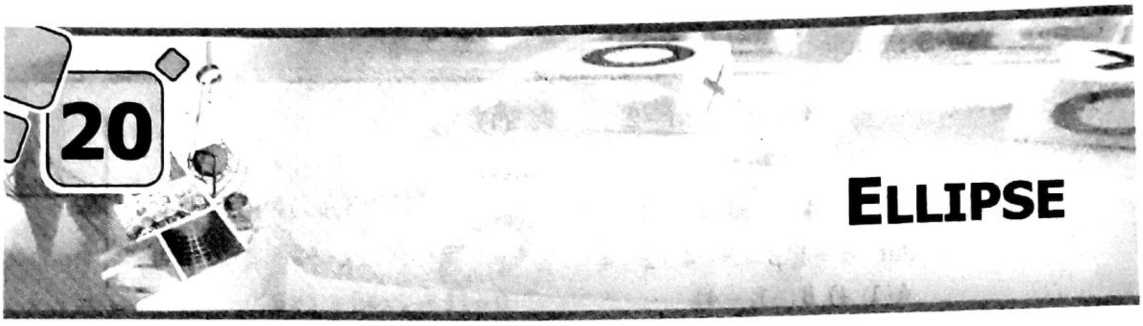
Chapter 20 – Ellipse



20

ELLIPSE

Exercise-1 : Single Choice Problems

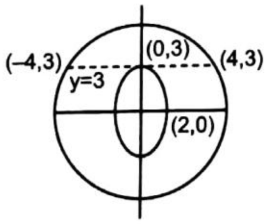
1. Length of perpendicular from $C(0,0)$ to the tangent at $P(2\sqrt{3}\cos\theta, 2\sqrt{2}\cos\theta)$ is

$$CF = \left|\frac{-1}{\sqrt{\dfrac{\cos^2\theta}{12}+\dfrac{\sin^2\theta}{8}}}\right|$$

Equation of normal at P is $\dfrac{2\sqrt{3}x}{\cos\theta}-\dfrac{2\sqrt{2}\,y}{\sin\theta}=12-8$ which meets the major axis at $G\left(\dfrac{2}{\sqrt{3}}\cos\theta, 0\right)$

$$CF \times PG = 8$$

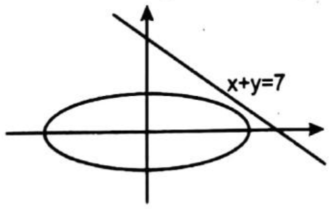
2.



The minimum length of intercept will be possible when

$$y=3 \quad \text{or} \quad y=-3 \Rightarrow AB=8$$

3.

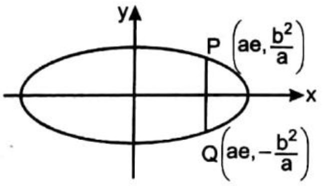


$$\frac{dy}{dx}=-\frac{x}{2y}=-1$$

Put $x=2y$ in the equation of ellipse

The point lies in I quad $\Rightarrow (2,1)$

4. Equation of tangent at P is


y
P $\left(ae,\frac{b^2}{a}\right)$
x
Q $\left(ae,-\frac{b^2}{a}\right)$

$$ex+y=a$$

$\Rightarrow$ $$e=\frac{2}{3},\quad a=\frac{10}{3}$$

and $$b=\frac{10\sqrt{5}}{9}$$

Length of latus rectum $=\dfrac{2b^2}{a}=\dfrac{100}{27}$

5. Area bounded by circle & ellipse $=\pi a^2-\pi ab=\pi a(a-b)$

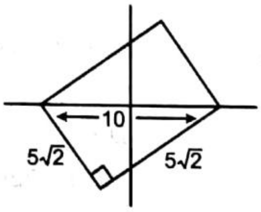
6. $\dfrac{S_1F_1+S_2F_2}{2}\geq\sqrt{(S_1F_1)(S_2F_2)}=\sqrt{16}$

$\because$ Product of perpendiculars from two foci of an ellipse upon any tangent is equal to the square of semi-minor axis.

7. $$f(k^2+2k+5)>f(k+11)$$

$\Rightarrow$ $$k^2+2k+5<k+11\ \Rightarrow\ k\in(-3,2)$$

8. Since sides of the square are tangent and perpendicular to each other, so the vertices lie on director circle



$$x^2+y^2=16+b^2=\left(\frac{10}{2}\right)^2$$

$\Rightarrow$ $$b=3$$

$$\frac{A}{\pi}=\frac{\pi(4)(3)}{\pi}=12$$

9. $T=S_1\quad\Rightarrow px+qy+\left(\dfrac{xq+py}{2}\right)-1=p^2+q^2+pq-1$

$$\Rightarrow p^2+q^2=-pq\ \Rightarrow\ p=0,q=0$$

10. The combined equation of pair of tangents drawn from a point (x_1,y_1) to the ellipse $S\equiv\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=0$ is $T^2=SS_1$. Therefore,

$$\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)$$

$$\left(\frac{4x}{9}+2y-1\right)^2=\left(\frac{x^2}{9}+y^2-1\right)\left(\frac{4^2}{9}+2^2-1\right)$$

$$\Rightarrow \quad 3x^2+7y^2-16xy+8x+36y-52=0$$

$$\Rightarrow \quad \tan\alpha=\frac{2\sqrt{h^2-ab}}{a+b}$$

where, $a=3, b=7$ and $h=-8$. Therefore,

$$\tan\alpha=\frac{2\sqrt{64-21}}{10}=\frac{\sqrt{43}}{5}$$

Note : α is acute angle between the pair of tangents. Therefore,

$$(a+b-c)^2=a^2+b^2+c^2+2ab-2ac-2bc$$

Alternate solution : Any line passing through the point (4, 2) is given by

$$y-2=m(x-4)$$

$$y=mx-4m+2$$

For this line to be tangent to the given ellipse, put this y into the equation of the ellipse and make

$$D=0$$

That is,

$$\frac{x^2}{9}+(mx-4m+2)^2=1$$

$$(1+9m^2)x^2+x(36m-72m^2)+16(9)m^2-16(9)m+27=0$$

Now,

$$D=0 \Rightarrow B^2-4AC=0$$

$$\Rightarrow \quad (36m-72m^2)^2-4(1+9m^2)(16\cdot 9m^2-16\cdot 9m+27)=0$$

$$\Rightarrow \quad (36m)^2(1-2m)^2-36(1+9m^2)(16m^2-16m+3)=0$$

$$\Rightarrow \quad m^2(1+4m^2-4m)-36(16m^2-16m+3+9\cdot 16m^4-9\cdot 16m+27m^2)=0$$

$$\Rightarrow \quad 7m^2-16m+3=0$$

Now,

$$\tan\alpha=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\left|\frac{\sqrt{(m_1+m_2)^2-4m_1m_2}}{1+m_1m_2}\right|$$

$$\Rightarrow \quad \tan\alpha=\frac{\sqrt{\left(\frac{16}{7}\right)^2-4\cdot\frac{3}{2}}}{1+\frac{3}{7}}=\frac{7}{10}\left(\frac{\sqrt{16^2-4\cdot 3\cdot 7}}{7}\right)$$

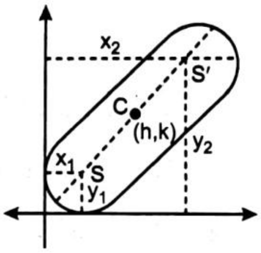
$$=\left(\frac{1}{10}\right)\sqrt{4(43)}=\frac{\sqrt{43}}{5}$$

where α is the acute angle between the tangents.

Exercise-2 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 2

1. $SS'=2ae$



$$(x_1-x_2)^2+(y_1-y_2)^2=4(2)^2\left(\frac{\sqrt{3}}{2}\right)^2$$

$$(x_1+x_2)^2+(y_1+y_2)^2-4(x_1x_2+y_1y_2)=12$$

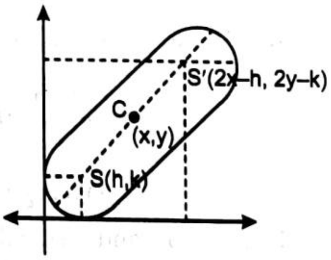
$$(2h)^2+(2k)^2-4(1+1)=12$$

($\because$ x_1x_2 and y_1y_2 are $\perp$ distance of the foci from their tangents $=b^2=1^2$)

$$\Rightarrow \qquad h^2+k^2=5$$

2. $(2x-h)(h)=1 \quad \Rightarrow \quad x=\dfrac{1+h^2}{2h}$

$(2y-k)(k)=1 \quad \Rightarrow \quad y=\dfrac{1+k^2}{2k}$



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Chapter 21 – Hyperbola

21

HYPERBOLA

Exercise-1 : Single Choice Problems

1. The normal is $y-4=\frac{1}{4}(x-1)$. Put the value of y in $xy=4$ we get co-ordinates.

3. $c^2=a^2m^2-b^2 \Rightarrow c^2=\lambda^2m^2-(\lambda^3+\lambda^2+\lambda)^2$

$c^2 \geq 0 \quad \Rightarrow \quad m^2 \geq (\lambda^2+\lambda+1)^2$

$\lambda^2+\lambda+1$ has minimum value $\frac{3}{4} \Rightarrow m^2 \geq \frac{9}{16}$

4. The asymptotes are $y=\pm\frac{\sqrt{3}}{2}x$ and the double ordinate be

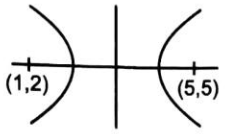
$$P\left(h, \frac{\sqrt{3}}{2}\sqrt{h^2-4}\right) \text{ and } P'\left(h, -\frac{\sqrt{3}}{2}\sqrt{h^2-4}\right)$$

$\Rightarrow \quad (PQ)(PQ')=3$

5. $2ae=5$ and $2a=3$

$\Rightarrow \quad e=\frac{5}{3}$

$\Rightarrow \quad \frac{1}{e^2}+\frac{1}{(e')^2}=1 \Rightarrow e'=\frac{5}{4}$



6. The equation of normal *at* $(2\sec\theta, \tan\theta)$ is $2x\cos\theta+y\cot\theta=5$

Equal intercepts $\quad \Rightarrow \quad \sin\theta=\frac{1}{2}$

Also touches ellipse $\quad \Rightarrow \quad a^2+b^2=\frac{25}{3} \quad \because \ c^2=a^2m^2+b^2$

7. Let locus of point be (h,k).

Equation of chord of contact is $hx+ky=4$

For tangent, $x\left(\frac{4-hx}{k}\right)=1$ has two equal roots.

$\Rightarrow \quad hk = 4 \Rightarrow xy = 4$

8. $\dfrac{x^2}{16} - \dfrac{y^2}{18} - \left(\dfrac{x\cos\alpha + y\sin\alpha}{p}\right)^2 = 0$

$\Rightarrow$ Coeff. of x^2 + coeff. of $y^2 = 0 \Rightarrow P = \pm 12$

The chord $x\cos\alpha + y\sin\alpha \pm 12 = 0$ is tangent to the circle $x^2 + y^2 = \left(\dfrac{d}{2}\right)^2 \Rightarrow \dfrac{d}{4} = 6$

9. Let the rectangular hyperbola be $x^2 - y^2 = a^2$ and the point be $(a\sec\theta, a\tan\theta)$.

$$a_1a_2 + b_1b_2 = (a\cos\theta)\left(\frac{2a}{\cos\theta}\right) + \left(-\frac{a\cos\theta}{\sin\theta}\right)\left(\frac{2a\sin\theta}{\cos\theta}\right)$$

Exercise-2 : One or More than One Answer is/are Correct

3. Let $\left(t, \dfrac{1}{t}\right)$ be any point on $xy = 1$

$$xy = 1$$

$$\Rightarrow \quad xy' + y = 0$$

$$\Rightarrow \quad y' = \frac{-y}{x}$$

$$\Rightarrow \quad y' = -\frac{1}{t^2}$$

$$\Rightarrow \quad \frac{-b}{a} = t^2$$

$\Rightarrow$ a and b are of opp. sign.

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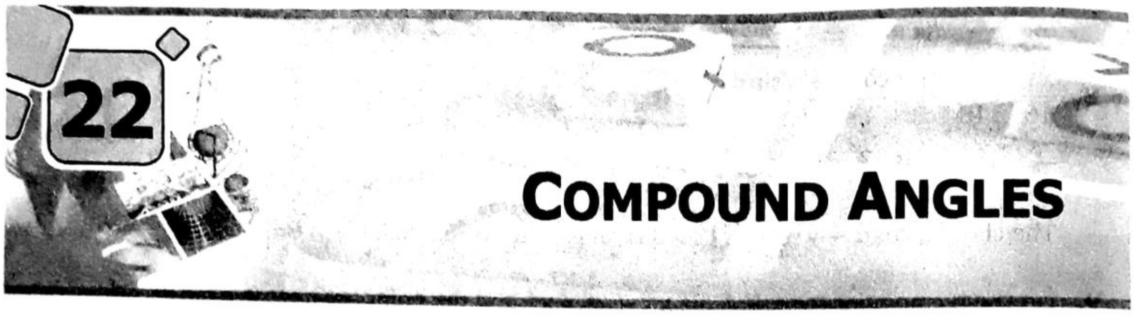
Chapter 22 – Compound Angles



22

COMPOUND ANGLES

Exercise-1 : Single Choice Problems

2.

$$a\sin x + b(2\cos c\cos x) = \alpha$$

$$\cos c = \frac{\alpha - a\sin x}{2b\cos x}$$

$$= \frac{1}{2b}(\alpha\sec x - a\tan x) \text{ differentiate w.r.t. } x$$

$$\alpha\sec x\tan x - a\sec^2 x = 0$$

$$\Rightarrow \quad \sin x = \frac{a}{\alpha}$$

3.

$$\tan x\cdot\frac{3\tan x - \tan^3 x}{1-3\tan^2 x} < -1$$

$$t\left(\frac{3t-t^3}{1-3t^2}\right)+1<0 \qquad (\text{Let } \tan x = t)$$

$$\frac{1-t^4}{1-3t^2}<0 \quad \Rightarrow \quad \frac{(t-1)(t+1)}{(3t^2-1)}<0$$

$$\Rightarrow \quad t\in\left(\frac{1}{\sqrt{3}},1\right)$$

4. $\displaystyle\sum_{r=1}^{8}\tan(rA)\tan\{(r+1)A\} = \sum_{r=1}^{8}\left[\frac{\tan(r+1)A-\tan(rA)-\tan A}{\tan A}\right] = \frac{\tan 9A - 9\tan A}{\tan A} = -10$

5.

$$f(x) = 2\operatorname{cosec} 2x + \sec x + \operatorname{cosec} x$$

$$= \frac{1+\sin x+\cos x}{\sin x\cos x}$$

$$f'(x) = \frac{\sin^3 x + \sin^2 x - \cos^3 x - \cos^2 x}{\sin^2 x\cos^2 x} = 0 \quad \Rightarrow \quad x = \frac{\pi}{4}$$

$$f(x)_{\min} = \frac{2}{\sqrt{2}-1} \qquad \text{at } x = \frac{\pi}{4}$$

6. $\operatorname{cosec}\theta + \operatorname{cosec}(60°-\theta) - \operatorname{cosec}(60°+\theta)$

where $\theta = 10°$

10. $\frac{1}{2}(2\sin x\cos x + 2\cos^2 x) = \frac{1}{2}(\sin 2x + \cos 2x + 1)$

11. $\frac{\tan A}{\sqrt{3}} = \frac{\tan B}{\sqrt{5}} = k \qquad (k > 0),\ \text{if } 2\sin A = \sqrt{3}\sin B$

$$\Rightarrow \frac{2\tan A}{\sqrt{1+\tan^2 A}} = \frac{\sqrt{3}\tan B}{\sqrt{1+\tan^2 B}} \Rightarrow \frac{2\sqrt{3}k}{\sqrt{1+3k^2}} = \frac{\sqrt{3}\times\sqrt{5}k}{\sqrt{1+5k^2}} \Rightarrow k = \frac{1}{\sqrt{5}}$$

12. Gives equations can be written as

$$2\cos\alpha + 9\cos\delta = -6\cos\beta - 7\cos\gamma \qquad \text{...(1)}$$

$$2\sin\alpha - 9\sin\delta = 6\sin\beta - 7\sin\gamma \qquad \text{...(2)}$$

Square and add equation (1) and (2),

$$\Rightarrow 4 + 36 + 36[\cos\alpha\cos\delta - \sin\delta\sin\alpha] = 36 + 49 + 84[\cos\beta\cos\gamma - \sin\beta\sin\gamma]$$

$$\Rightarrow 36[\cos(\alpha+\delta)] = 84[\cos(\beta+\gamma)]$$

$$\frac{\cos(\alpha+\delta)}{\cos(\beta+\gamma)} = \frac{84}{36} = \frac{7}{3} = \frac{m}{n}; \qquad m + n = 10$$

13. $\left|\frac{1+\sin\theta+1-\sin\theta}{\sqrt{1-\sin^2\theta}}\right| = \left|\frac{2}{|\cos\theta|}\right| = -2\sec\theta$

14. $A = \sum_{r=1}^{3}\cos\frac{2r\pi}{7} = \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7} = B$

15. $$\tan\beta = \frac{x}{z} = \frac{1}{3}$$

$$\tan\alpha = \frac{y}{z} = \frac{1}{2}$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} = 1 \quad \Rightarrow \quad \alpha + \beta = \frac{\pi}{4}$$

17. $f(x) = -2\sin^2 x + \sin x + 2\ \forall\ x \in \left[\frac{\pi}{6}, \frac{2\pi}{3}\right]$

Let $\sin x = t$

$f(t) = -2t^2 + t + 2\ \forall\ t \in \left[\frac{1}{2}, 1\right]$

18. $1+(\cos^2 A-\sin^2 B)-\cos A\cos B=1+\cos(A+B)\cos(A-B)-\frac{1}{2}[\cos(A+B)+\cos(A-B)]=\frac{3}{4}$

19. $(2\sin x-\operatorname{cosec} x)^2+(\tan x-\cot x)^2=0$

$$\therefore\quad \sin^2 x=\frac{1}{2}\ \cap\ \tan^2 x=1$$

20. $\cos^2 A=\sin A\cdot\tan A\quad\Rightarrow\quad\cos^3 A=\sin^2 A$

21. $f(x)=\left(\frac{\sqrt{3}+1}{2}\right)\sin x+\left(\frac{\sqrt{3}+1}{2}\right)\cos x=\left(\frac{\sqrt{3}+1}{2}\right)(\sin x+\cos x)$

22. $A=B+C$

$$\Rightarrow\quad \tan A\tan B\tan C=\tan A-\tan B-\tan C$$

23.

$$E=\sin A+\sin 2B+\sin 3C$$

$$E=\frac{3}{5}+2\cdot\frac{4}{5}\cdot\frac{3}{5}-1$$

$$=\frac{15}{25}+\frac{24}{25}-1=\frac{39-25}{25}=\frac{14}{25}$$

24. $\dfrac{\cos A\cos C+\cos A\cos C}{\cos A\sin C+\cos A\sin C}=\cot C\qquad(\because\ A+B+C=\pi)$

25. $\dfrac{\sin\alpha-\sin\gamma}{\cos\gamma-\cos\alpha}=\dfrac{2\cos\left(\frac{\alpha+\gamma}{2}\right)\sin\left(\frac{\alpha-\gamma}{2}\right)}{2\sin\left(\frac{\alpha+\gamma}{2}\right)\sin\left(\frac{\alpha-\gamma}{2}\right)}=\cot\left(\frac{\alpha+\gamma}{2}\right)=\cot\beta$

26. $\cos\frac{x}{256}\cdot\cos\frac{x}{128}\cos\frac{x}{64}\cdots\cdots\cos\frac{x}{4}\cdot\cos\frac{x}{2}=\dfrac{\sin x}{256\sin\left(\frac{x}{256}\right)}$

27. $\dfrac{(\sin 7\alpha+\sin 5\alpha)+5(\sin 5\alpha+\sin 3\alpha)+12(\sin 3\alpha+\sin\alpha)}{\sin 6\alpha+5\sin 4\alpha+12\sin 2\alpha}$

$$=\frac{2\sin 6\alpha\cos\alpha+5(2\sin 4\alpha\cos\alpha)+12(2\sin 2\alpha\cos\alpha)}{\sin 6\alpha+5\sin 4\alpha+12\sin 2\alpha}=2\cos\alpha$$

28. $\tan^2 A+\tan^2 B+\tan^2 C=\tan A\tan B+\tan B\tan C+\tan A\tan C$

$$\Rightarrow\quad \tan A=\tan B=\tan C$$

$$\Rightarrow\quad A=B=C=\frac{\pi}{3}$$

29. $\log_{|\sin x|}|\cos x|+\log_{|\cos x|}|\sin x|=2\quad\Rightarrow\quad\log_{|\sin x|}|\cos x|=1\ \Rightarrow\ |\cos x|=|\sin x|$

30. $f(x)=\sin^6 x+\cos^6 x=1-3\sin^2 x\cos^2 x=1-\frac{3}{4}\sin^2 2x$

31. $$y = \frac{2\sin\alpha}{1+\cos\alpha+\sin\alpha} \times \left[\frac{(1+\sin\alpha)-\cos\alpha}{(1+\sin\alpha)-\cos\alpha}\right] = \frac{2\sin\alpha[(1+\sin\alpha)-\cos\alpha]}{(1+\sin\alpha)^2-\cos^2\alpha}$$

$$= \frac{1+\sin\alpha-\cos\alpha}{1+\sin\alpha}$$

32. $$\frac{\tan^3 A}{1+\tan^2 A} + \frac{\cot^3 A}{1+\cot^2 A} = \frac{\sin^3 A}{\cos A} + \frac{\cos^3 A}{\sin A}$$

$$= \frac{\sin^4 A+\cos^4 A}{\sin A\cos A} = \frac{1-2\sin^2 A\cos^2 A}{\sin A\cos A}$$

$$= \sec A \operatorname{cosec} A - 2\sin A\cos A$$

33. $$\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} + \sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \frac{2}{\sqrt{1-\sin^2\theta}} = \frac{2}{|\cos\theta|}$$

34. $y = (\sin^2\theta + \operatorname{cosec}^2\theta + 2) + (\cos^2\theta + \sec^2\theta + 2) = 7 + (\tan^2\theta + \cot^2\theta) \geq 9$

35. $\log_3 \sin x - \log_3 \cos x - \log_3 (1-\tan x) - \log_3 (1+\tan x) = -1$

$$\log_3\left(\frac{\tan x}{1-\tan^2 x}\right) = -1 \quad \Rightarrow \quad \frac{\tan x}{1-\tan^2 x} = \frac{1}{3} \quad \Rightarrow \quad \tan 2x = \frac{2}{3}$$

36. $\sin\theta + \operatorname{cosec}\theta = 2 \quad \Rightarrow \quad \sin\theta = \operatorname{cosec}\theta = 1; \quad \left(x + \frac{1}{x} \geq 2\right)$

37. $(\tan\theta + \cot\theta)(\tan^2\theta + \cot^2\theta - 1) = 52$

$(\tan\theta + \cot\theta)\{(\tan\theta + \cot\theta)^2 - 3\} = 52$

Let $\tan\theta + \cot\theta = t$

$$t^3 - 3t - 52 = 0 \quad \Rightarrow \quad t = 4$$

$$\tan^2\theta + \cot^2\theta = (\tan\theta + \cot\theta)^2 - 2 = 14$$

38. $$-5 \leq 3\sin x - 4\cos x \leq 5$$

$$10 \leq 3\sin x - 4\cos x + 15 \leq 20$$

$$\log_{20} 10 \leq \log_{20}(3\sin x - 4\cos x + 15) \leq \log_{20} 20$$

39. $$x^2 + y^2 = 9$$

Let $x = 3\cos\theta,\ y = 3\sin\theta$

$$4a^2 + 9b^2 = 16$$

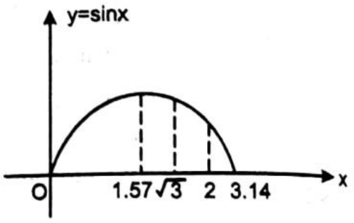
Let $a = 2\cos\phi,\ b = \frac{4}{3}\sin\phi$

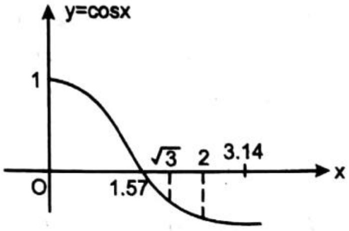
$$4a^2x^2 + 9b^2y^2 - 12abxy = (2ax - 3by)^2$$

$$= (12\cos\theta\cos\phi - 12\sin\theta\sin\phi)^2 = 144\cos^2(\theta + \phi)$$

40. $A^2 = \sin 2 - \sin\sqrt{3}$

$$\sin\sqrt{3} > \sin 2$$
$$A^2 < 0$$
$$B^2 = \cos 2 - \cos\sqrt{3}$$
$$\cos\sqrt{3} > \cos 2$$
$$B^2 < 0$$





Both A and B are not real numbers.

41. $(2^x + 2^{-x} - 2\cos x)(3^{x+\pi} + 3^{-x-\pi} + 2\cos x)(5^{\pi-x} + 5^{x-\pi} - 2\cos x) = 0$

If $\dfrac{2^x + 2^{-x}}{2} = \cos x \quad \Rightarrow \quad x = 0$

If $\dfrac{3^{x+\pi} + 3^{-x-\pi}}{2} = -\cos x \quad \Rightarrow \quad x = -\pi$

If $\dfrac{5^{\pi-x} + 5^{x-\pi}}{2} = \cos x$ (Not possible)

There are two real values of x.

42.
$$e^{\sin x} - e^{-\sin x} - 4 = 0$$
$$e^{2\sin x} - 4e^{\sin x} - 1 = 0$$

$\Rightarrow \quad e^{\sin x} = 2 \pm \sqrt{5}$

If $\quad e^{\sin x} = 2 + \sqrt{5}$

$\Rightarrow \quad \sin x = \ln(2 + \sqrt{5})$ [$\ln(2 + \sqrt{5}) > 1$, Not possible]

If $\quad e^{\sin x} = 2 - \sqrt{5}$ ($2 - \sqrt{5} < 0$) Not possible

There is no solution.

43. $\sqrt{4\sin^4\alpha + 4\sin^2\alpha \cdot \cos^2\alpha} + 4\cos^2(\pi/4 - \alpha/2)$

$$= \sqrt{4\sin^2\alpha} + 2[1 + \cos(\pi/2 - \alpha)]$$
$$= 2|\sin\alpha| + 2 + 2\sin\alpha$$
$$= -2\sin\alpha + 2 + 2\sin\alpha = 2$$

(If $\pi < \alpha < \dfrac{3\pi}{2}$ then $\sin\alpha < 0$)

44. $\left(\cos\frac{\pi}{12}-\sin\frac{\pi}{12}\right)\left(\frac{\sin\frac{\pi}{12}}{\cos\frac{\pi}{12}}+\frac{\cos\frac{\pi}{12}}{\sin\frac{\pi}{12}}\right)$

$$=\frac{\cos\frac{\pi}{12}-\sin\frac{\pi}{12}}{\sin\frac{\pi}{12}\cdot\cos\frac{\pi}{12}}=\frac{2\sqrt{1-\sin\pi/6}}{\sin\pi/6}=2\sqrt{2}$$

45. $\tan(100°+125°)=\frac{\tan 100°+\tan 125°}{1-\tan 100°\tan 125°}=1$

$\Rightarrow$ $\tan 100°+\tan 125°+\tan 100°\tan 125°=1$

46. If $\sin x+\sin^2 x=1 \Rightarrow \sin x=1-\sin^2 x \Rightarrow \sin x=\cos^2 x$

$$\cos^8 x+2\cos^6 x+\cos^4 x=\sin^4 x+2\sin^3 x+\sin^2 x$$
$$=\sin^2 x(\sin^2 x+2\sin x+1)$$
$$=(1-\sin x)(2+\sin x)$$
$$=2-\sin x-\sin^2 x=1$$

47. Let $x=5\cos\theta$, $y=5\sin\theta$

$$0<3x+4y\le 25 \qquad (\because\ 3x+4y>0)$$

48. $5\cos 2\theta+2\cos^2\left(\frac{\theta}{2}\right)+1=0$

$$10\cos^2\theta+\cos\theta-3=0 \quad\Rightarrow\quad \cos\theta=\frac{1}{2},\ -\frac{3}{5}$$

49. $\sin\beta=\frac{4}{5}$ where $0<\beta<\pi$ and $\tan\beta>0$

then $\cos\beta=\frac{3}{5}$

$$5\left[\frac{3}{5}\sin(\alpha+\beta)-\frac{4}{5}\cos(\alpha+\beta)\right]\operatorname{cosec}\alpha=5$$

50. $\sin\left(x+\frac{\pi}{6}\right)+\cos\left(x+\frac{\pi}{6}\right)=\sqrt{2}\left[\cos\frac{\pi}{4}\sin\left(x-\frac{\pi}{6}\right)+\sin\frac{\pi}{4}\cdot\cos\left(x+\frac{\pi}{6}\right)\right]=\sqrt{2}\sin\left(x+\frac{5\pi}{12}\right)$

This attained maximum value when $x+\frac{5\pi}{12}=\frac{\pi}{2} \Rightarrow x=\frac{\pi}{12}$

51.
$$\sin 2x-\cos 2x=2a-1$$
$$-\sqrt{2}\le 2a-1\le\sqrt{2}$$
$$\frac{1-\sqrt{2}}{2}\le a\le\frac{1+\sqrt{2}}{2}$$

52.
$$(\cos 12°\cdot\cos 24°\cdot\cos 48°\cdot\cos 84°)(\cos 36°\cos 72°)\cdot\cos 60°$$
$$(-\cos 12°\cdot\cos 24°\cdot\cos 48°\cdot\cos 96°)(\cos 36°\cos 72°)\cdot\cos 60°$$

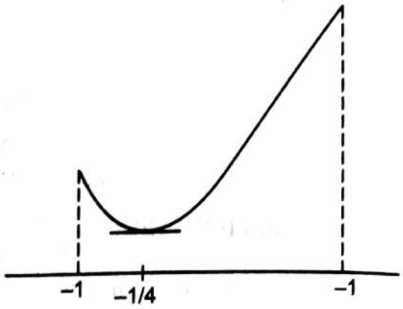
$$\left[-\frac{\sin(2^4\times 12^\circ)}{2^4\sin 12^\circ}\right]\times\left(\frac{\sqrt{5}+1}{4}\times\frac{\sqrt{5}-1}{4}\right)\times\frac{1}{2}=\frac{1}{128}$$

53. $2\cos^2\theta+\cos\theta+1$

$y_{\min}=\frac{7}{8}$ at $\cos\theta=-\frac{1}{4}$

$y_{\max}=4$ at $\cos\theta=1$

$$\frac{y_{\max}}{y_{\min}}=\frac{32}{7}$$

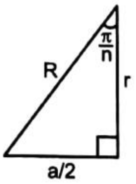


54. $\tan x\left(\frac{3\tan x-\tan^3 x}{1-3\tan^2 x}\right)+1<0;\ \frac{\tan^4 x-1}{3\tan^2 x-1}<0$

$$\Rightarrow\quad \frac{(\tan^2 x+1)(\tan x+1)(\tan x-1)}{(\sqrt{3}\tan x+1)(\sqrt{3}\tan x-1)}<0$$

$$\Rightarrow\quad \frac{\pi}{6}<x<\frac{\pi}{4}$$

55. $a=2R\sin\frac{\pi}{n}=2r\tan\frac{\pi}{n}$



56. $(\cos 12^\circ+\cos 132^\circ)+(\cos 84^\circ+\cos 156^\circ)$

$$=2\cos\frac{12^\circ+132^\circ}{2}\cos\frac{12^\circ-132^\circ}{2}+2\cos\frac{84^\circ+156^\circ}{2}\cos\frac{84^\circ-156^\circ}{2}$$

$$=2\cos 72^\circ\cos 60^\circ+2\cos 120^\circ\cos 36^\circ$$

$$=2\times\frac{\sqrt{5}-1}{4}\times\frac{1}{2}+2\times\left(-\frac{1}{2}\right)\times\frac{\sqrt{5}+1}{4}=-\frac{1}{2}$$

57. $\frac{1}{2}\left[\frac{2\sin\theta\cos\theta}{\cos\theta\cos 3\theta}+\frac{2\sin 3\theta\cos 3\theta}{\cos 9\theta\cos 3\theta}+\frac{2\sin 9\theta\cos 9\theta}{\cos 9\theta\cos 27\theta}+\frac{2\sin 27\theta\cos 27\theta}{\cos 27\theta\cos 81\theta}\right]$

$$=\frac{1}{2}\left[\frac{\sin(3\theta-\theta)}{\cos\theta\cos 3\theta}+\frac{\sin(9\theta-3\theta)}{\cos 3\theta\cos 9\theta}+\frac{\sin(27\theta-9\theta)}{\cos 9\theta\cos 27\theta}+\frac{\sin(81\theta-27\theta)}{\cos 27\theta\cos 81\theta}\right]$$

$$=\frac{1}{2}[\tan 81\theta-\tan\theta]=\frac{1}{2}\left[\frac{\sin 80\theta}{\cos\theta\cos 81\theta}\right]$$

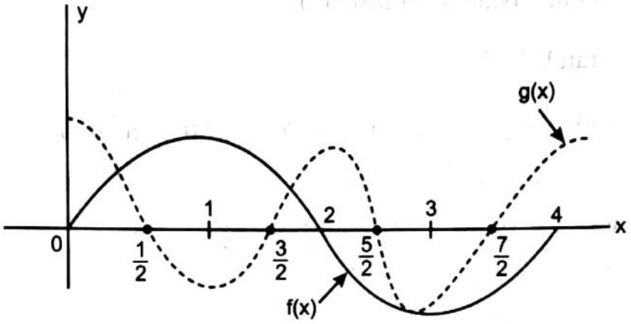
58. $\sin 20°\left(\dfrac{4\cos 20°+1}{\cos 20°}\right)=\dfrac{2\sin 40°+\sin 20°}{\cos 20°}=\dfrac{2\sin(60°-20°)+\sin 20°}{\cos 20°}=\sqrt{3}$

59. Let us draw the graph of

$$f(x)=\sin\left(\frac{x\pi}{2}\right)$$

and $$g(x)=\cos(x\pi)$$

On the same xy-plane as shown in the following figure.



From this graphical representation, it is clear that y is strictly increasing in $\left(\dfrac{5}{2},\dfrac{7}{2}\right)$

Because for all values of x,

$$\frac{5}{2}<x<\frac{7}{2}$$

That is, $$\sin\left(\frac{x\pi}{2}\right)<0$$

and $$\cos(x\pi)<0$$

which imply that $$\frac{dy}{dx}>0$$

which means that y is strictly increasing.

60. $8\sin\theta\sin 3\theta\left(\dfrac{\sin 8\theta}{4\sin 2\theta}\right)=\cos 6\theta$

$$\sin 3\theta\sin 8\theta=\cos 6\theta\cos\theta$$
$$\cos 5\theta-\cos 11\theta=\cos 7\theta+\cos 5\theta$$
$$\cos 7\theta+\cos 11\theta=0$$
$$2\cos 9\theta\cdot\cos 2\theta=0$$

61. $\tan A=-\dfrac{1}{3}\Rightarrow\sin A=\dfrac{1}{\sqrt{10}};\quad\cos A=-\dfrac{3}{\sqrt{10}}$

63. $(2\cos\theta)^2=(1-\sin\theta)^2\ \Rightarrow\ \sin\theta=1\text{ or }\sin\theta=\dfrac{-3}{5}$

64. $\sin\theta + \dfrac{1}{\sin\theta} = 2 \Rightarrow \sin\theta = 1$

65. $\tan^2\theta + \cot^2\theta = a \Rightarrow \tan^3\theta + \cot^3\theta = \sqrt{a+2}\,(a-1) = 52$

66. $\tan A = -\tan C = \dfrac{5}{12}$

$\cos B = -\cos D = -\dfrac{3}{5} \quad \Rightarrow \tan D = \dfrac{4}{3}$

67. $\sqrt{\tan^2\theta - \sin^2\theta} = \sqrt{\tan^2\theta\sin^2\theta} = |\tan\theta\sin\theta|$

68. $\dfrac{\sin 10° + \sin 20°}{\cos 10° + \cos 20°} = \tan 15° = 2 - \sqrt{3}$

69. $(\sin^2\theta)^3 + (\cos^2\theta)^3 = (\sin^2\theta + \cos^2\theta)(\sin^4\theta + \cos^4\theta - \sin^2\theta\cos^2\theta)$

$$= 1 - 3\sin^2\theta\cos^2\theta$$

70. $\dfrac{\tan x + 1}{\tan x - 1} - \dfrac{\sec^2 x + 2}{\tan^2 x - 1} \Rightarrow \dfrac{(\tan x + 1)^2 - (\sec^2 x + 2)}{\tan^2 x - 1}$

$$\Rightarrow \frac{2\tan x - 2}{\tan^2 x - 1} \Rightarrow \frac{2}{\tan x + 1}$$

71. $\dfrac{\cot\alpha - \tan\alpha}{\cot\alpha + \tan\alpha} - [\cos 450° + \cos(2\alpha - 180°)]$

$\Rightarrow \quad (\cos^2\alpha - \sin^2\alpha) + \cos 2\alpha = 2\cos 2\alpha$

72. $\left(\dfrac{1+\tan\alpha}{1-\tan\alpha}\right)\cdot\left(\dfrac{1+\tan\alpha}{1-\tan\alpha}\right) + 1$

$1 + \tan^2\left(\dfrac{\pi}{4} + \alpha\right) = \sec^2\left(\dfrac{\pi}{4} + \alpha\right) = \operatorname{cosec}^2\left(\dfrac{\pi}{4} - \alpha\right)$

73. $\dfrac{\tan\alpha + \sin\alpha}{1 + \cos\alpha} = \tan\alpha$

74. $(\cos 2\alpha + \cos 5\alpha) - (\cos 3\alpha + \cos 4\alpha)$

$2\cos\dfrac{7\alpha}{2}\cdot\cos\dfrac{3\alpha}{2} - 2\cos\dfrac{7\alpha}{2}\cdot\cos\dfrac{\alpha}{2}$

$2\cos\dfrac{7\alpha}{2}\left[\cos\dfrac{3\alpha}{2} - \cos\dfrac{\alpha}{2}\right] = -4\sin\dfrac{\alpha}{2}\sin\alpha\cos\dfrac{7\alpha}{2}$

75. $\cos 2\gamma = \dfrac{1 - \tan^2\gamma}{1 + \tan^2\gamma} = \dfrac{1 - \left(\dfrac{1 + \sin\alpha\sin\beta}{\cos\alpha\cos\beta}\right)^2}{1 + \left(\dfrac{1 + \sin\alpha\sin\beta}{\cos\alpha\cos\beta}\right)^2}$

$$\Rightarrow \quad \frac{(\cos\alpha\cos\beta)^2 - (1 + \sin\alpha\cos\beta)^2}{(\cos\alpha\cos\beta)^2 + (1 + \sin\alpha\sin\beta)^2} = \frac{[1 + \cos(\alpha - \beta)][\cos(\alpha + \beta) - 1]}{(\cos\alpha\cos\beta)^2 + (1 + \sin\alpha\sin\beta)^2} \le 0$$

76. $x = \frac{2\pi}{3}$ (II$^{\text{nd}}$ quadrant)

$$\cos x + \cos 2x + \cos 3x + \ldots + \cos 100x = \frac{\sin 50x}{\sin \frac{x}{2}} \cdot \cos\left(\frac{101x}{2}\right) = -\frac{1}{2}$$

77. $\cos^3 0° + \cos^3 \frac{\pi}{3} + \cos^3 \frac{2\pi}{3} + \cos^3 \pi + \ldots + \cos^3 \frac{10\pi}{3} = -\frac{1}{8}$

78. $\frac{1 - 2(\cos 60° - \cos 80°)}{2 \sin 10°} = \frac{2 \cos 80°}{2 \sin 10°} = 1$

79. $(x+5)^2 + (y-12)^2 = 14^2$

Let $x = -5 + 14\cos\theta,\ y = 12 + 14\sin\theta$

$\Rightarrow \quad x^2 + y^2 = 365 + 336\sin\theta - 140\cos\theta$

80. $\tan\theta = \lambda$ has three distinct solution in $[0, 2\pi] \Rightarrow \lambda = 0$ and $\theta = 0, \pi, 2\pi$.

81. $\sqrt{\frac{1+\tan\alpha}{1-\tan\alpha}} + \sqrt{\frac{1-\tan\alpha}{1+\tan\alpha}} = \frac{2}{\sqrt{1-\tan^2\alpha}}$

82. $3\sin\theta + 4\cos\theta = 5\left(\frac{3}{5}\sin\theta + \frac{4}{5}\cos\theta\right) = 5\sin(\theta + 53°)$

83. $f(n) = \prod_{r=1}^{n} \cos r$

$f(4) = \cos 1 \cdot \cos 2 \cdot \cos 3 \cdot \cos 4 < 0$

$f(5) = \cos 1 \cdot \cos 2 \cdot \cos 3 \cdot \cos 4 \cdot \cos 5 < 0$

84. $\frac{(p^2 - q^2)^2}{pq} = \frac{(4\tan A \sin A)^2}{\tan^2 A - \sin^2 A} = 16$

85. $0 < \sin\alpha < \cos\alpha < 1 \qquad \alpha \in \left(0, \frac{\pi}{4}\right)$

$(\sin\alpha)^{\cos\alpha} < (\sin\alpha)^{\sin\alpha}$

$(\cos\alpha)^{\cos\alpha} < (\cos\alpha)^{\sin\alpha}$

86. $32\sin\frac{A}{2}\sin\frac{5A}{2} = 16(\cos 2A - \cos 3A)$

$$= 16[(2\cos^2 A - 1) - (4\cos^3 A - 3\cos A)]$$

87. $\cos\alpha\cos\beta - \sin\alpha\sin\beta + \sin\alpha\cos\beta - \cos\alpha\sin\beta = 0$

$\cos\alpha(\cos\beta - \sin\beta) + \sin\alpha(\cos\beta - \sin\beta) = 0$

$(\cos\beta - \sin\beta)(\cos\alpha + \sin\alpha) = 0$

$\cos\alpha = -\sin\alpha \qquad (\because \cos\beta \neq \sin\beta)$

$\tan\alpha = -1$

88. $2^x = 3^y = 6^{-z} = k$

$x = \log_2 k, y = \log_3 k, z = -\log_6 k$

$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$

89. $(\sin\alpha + \sin\beta)^2 + (\cos\alpha + \cos\beta)^2 = \left(\frac{-21}{65}\right)^2 + \left(\frac{-27}{65}\right)^2$

$2 + 2\cos(\alpha - \beta) = \frac{1170}{(65)^2} = 4\cos^2\left(\frac{\alpha - \beta}{2}\right)$

90. $\mu^2 = a^2 + b^2 + 2\sqrt{(a^2\cos^2\theta + b^2\sin^2\theta)(a^2\sin^2\theta + b^2\cos^2\theta)}$

$= a^2 + b^2 + 2\sqrt{a^2b^2 + (a^4 + b^4 - 2a^2b^2)\sin^2\theta\cos^2\theta}$

91. $Q = \sum_{r=0}^{n} \frac{\sin(3^r\theta)\cos(3^r\theta)}{\cos(3^r\theta)\cos(3^{r+1}\theta)} = \frac{1}{2}\sum_{r=0}^{n} \tan(3^{r+1}\theta) - \tan(3^r\theta) = \frac{1}{2}P$

92. When $270° < \theta < 360°$, we have

$$\sqrt{2(1+\cos\theta)} = \sqrt{\left(2\cos^2\frac{\theta}{2}\right)}$$

which is non-negative. Now, the above equation can be written as

$$\sqrt{2(1+\cos\theta)} = 2\left|\cos\frac{\theta}{2}\right|$$

$$= -2\cos\frac{\theta}{2} \qquad \left(\because \cos\frac{\theta}{2} < 0 \text{ when } 135° < \frac{\theta}{2} < 180°\right)$$

Now, let us consider that $\sqrt{2 + \sqrt{2(1+\cos\theta)}}$

which is not-negative. That is,

$$\sqrt{2 + \sqrt{2(1+\cos\theta)}} = \sqrt{2 - 2\cos\frac{\theta}{2}}$$

$$= \sqrt{2}\sqrt{1 - \cos\frac{\theta}{2}} = \sqrt{2}\sqrt{2\sin^2\frac{\theta}{4}}$$

$$= 2\left|\sin\frac{\theta}{4}\right|$$

$$= 2\sin\frac{\theta}{4} \qquad \left(\because \sin\frac{\theta}{4} > 0 \text{ when } \frac{135°}{2} < \frac{\theta}{4} < 90°\right)$$

93. We know that $-\sqrt{2} \le \sin x + \cos x \le \sqrt{2}$

When $x = -\frac{3\pi}{4}$, we have $\sin x + \cos x = -\sqrt{2}$

when $x = -\frac{3\pi}{4}$, we have $y = -\sqrt{2} + 1 < 0$

which implies that options (1) and (2) are incorrect.

Now, at $x = \frac{\pi}{4}$, we have $\sin x + \cos x = \sqrt{2}$

That is, $(\sin 4x + \cos 4x)^2 \neq 2$. Therefore, $y \neq \sqrt{2} + 2$ for any $x \in R$.

which implies that option (4) is incorrect.

Note : The maximum value of $\sin x + \cos x$ is $\sqrt{2}$, for $x = \frac{\pi}{4}$ and the maximum value of $(\sin 4x + \cos 4x)^2$ is 2, for $x = \frac{\pi}{16}$.

94. $(\cos x + \cos y)^2 + (\sin x + \sin y)^2 = (-\cos z)^2 + (-\sin z)^2$

$$2 + 2\cos(x - y) = 1$$

95. $$\frac{1}{\sin 10°} + \frac{1}{\sin 50°} - \frac{1}{\sin 70°} = \frac{\sin 50° \sin 70° + \sin 10° \sin 70° - \sin 10° \sin 50°}{\sin 10° \sin 50° \sin 70°}$$

$$= \frac{\frac{1}{2}(\cos 20° - \cos 120° + \cos 60° - \cos 80° - \cos 40° + \cos 60°)}{\frac{1}{4}\sin 30°}$$

$$= \frac{\frac{1}{2}\left(\frac{3}{2} + \cos 20° - 2\cos 60° \cos 20°\right)}{\frac{1}{4}\sin 30°} = 6$$

Exercise-2 : One or More than One Answer is/are Correct

1. $\cot 12° \cot 24° \cot 48° [\cot 28° \cot(60°-28°) \cot(60°+28°)] = (\cot 12° \cot 48°)(\cot 24° \cot 84°)$

$$= \frac{\cot 36°}{\cot 72°} \times \frac{\cot 72°}{\cot 36°} = 1$$

2. $$\cot^4 x - 2(1 + \cot^2 x) + a^2 = 0$$

$\Rightarrow$ $$\cot^4 x - 2\cot^2 x + a^2 - 2 = 0$$

$\Rightarrow$ $$(\cot^2 x - 1)^2 = 3 - a^2$$

to have atleast one solution

$$3 - a^2 \geq 0$$

$\Rightarrow$ $$a^2 - 3 \leq 0$$

$$a \in [-\sqrt{3}, \sqrt{3}]$$

Integral values –1, 0, 1

$\therefore$ $$\text{Sum} = 0$$

3. (A) $$\tan 1 > \tan^{-1} 1 \Rightarrow \tan 1 > \frac{\pi}{4}$$

(B) $$\sin 1 > \cos 1$$
$$\sin 57.3° > \cos 57.3°$$

(C) $$\tan 1 < \sin 1 \qquad \text{(not possible)}$$

Because $\tan 57.3 > 1 > \sin 57.3°$

(D) $$\cos 1 < \frac{\pi}{4}$$

$\Rightarrow$ $$\cos(\cos 1) > \cos\left(\frac{\pi}{4}\right)$$

4. (A) $\tan 1 > 1$ and $\sin 1 < 1$, then $\log_{\sin 1} \tan 1 < 0$

(B) $1 + \tan 3 < 1$ and $\cos 1 < 1$, then $\log_{\cos 1}(1 + \tan 3) > 0$

(C) $\cos\theta + \sec\theta > 2$ and $\log_{10} 5 < 1$, then $\log_{\log 10} 5(\cos\theta + \sec\theta) < 0$

(D) $2\sin 18° < 1$ and $\tan 15° < 1$, then $\log_{\tan 15°} 2\sin 18° > 0$

5. Put $\sin\alpha = \dfrac{2\tan\left(\frac{\alpha}{2}\right)}{1+\tan^2\left(\frac{\alpha}{2}\right)}$, $\cos\alpha = \dfrac{1-\tan^2\left(\frac{\alpha}{2}\right)}{1+\tan^2\left(\frac{\alpha}{2}\right)}$

6. Given $\dfrac{\sin(2\alpha+\beta)}{\sin\beta} = \dfrac{3}{1}$

Option (C) $\dfrac{\sin(2\alpha+\beta)+\sin\beta}{\sin(2\alpha+\beta)-\sin\beta} = \dfrac{3+1}{3-1}$ (Use C and D method)

$$\tan(\alpha+\beta) = 2\tan\alpha$$

Option (B) $$3\sin\beta = \sin(2\alpha+\beta)$$
$$2\sin\beta = \sin(2\alpha+\beta) - \sin\beta$$
$$2\sin\beta = 2\cos(\alpha+\beta)\sin\alpha$$

Option (D) $$3\sin\beta = \sin\{\alpha + (\alpha+\beta)\}$$
$$3\sin\beta = \sin\alpha\cos(\alpha+\beta) + \cos\alpha\sin(\alpha+\beta)$$

Subtract from (B) option
$$2\sin\beta = \cos\alpha\sin(\alpha+\beta)$$

Option (A) $\cot\beta - 3\cot(2\alpha+\beta) = \dfrac{\cos\beta}{\sin\beta} - 3\dfrac{\cos(2\alpha+\beta)}{\sin(2\alpha+\beta)}$

$$= \frac{\cos\beta}{\sin\beta} - 3\frac{\cos(2\alpha+\beta)}{3\sin\beta} = \frac{2\sin(\alpha+\beta)\sin\alpha}{\sin\beta} = 4\tan\alpha \quad \text{(from D)}$$

Also $$\cot\alpha + \cot(\alpha+\beta) = \frac{3}{2}\cot\alpha \quad \text{(from C)}$$

Now multiply the two relations.

7. $$\sin(x+20^\circ) = \sin(x+40^\circ) + \sin(x-40^\circ)$$

$$\sin(x+20^\circ) - \sin(x-40^\circ) = \sin(x+40^\circ)$$

$$\cos(x-10^\circ) = \sin(x+40^\circ) = \cos[90^\circ-(x+40^\circ)]$$

$\Rightarrow$ $x = 30^\circ$ now check the option, only (a) and (b) satisfy

8. $$2\Sigma(\cos x\cos y) + 2\Sigma(\sin x\sin y) + 3 = 0$$

$$(\Sigma\cos x)^2 + (\Sigma\sin x)^2 = 0$$

$\Rightarrow$ $\Sigma\cos x = 0$ and $\Sigma\sin x = 0$

$$\cos 3x + \cos 3y + \cos 3z = 4(\cos^3 x + \cos^3 y + \cos^3 z) - 3(\cos x + \cos y + \cos z)$$

$$= 12\cos x\cos y\cos z$$

9. $0 < \sin x < 1,\ 0 < \cos x < 1$

If $\sin^n x + \cos^n x = 1$ $\quad n = 2$

$\sin^n x + \cos^n x > 1$ $\quad n < 2$

$\sin^n x + \cos^n x < 1$ $\quad n > 2$

10. If $$x = \sin(\alpha-\beta)\sin(\gamma-\delta)$$

$$2x = \cos(\alpha-\beta-\gamma+\delta) - \cos(\alpha-\beta+\gamma-\delta)$$

$$y = \sin(\beta-\gamma)\sin(\alpha-\delta)$$

$\Rightarrow$ $$2y = \cos(\beta-\gamma-\alpha+\delta) - \cos(\beta-\gamma+\alpha-\delta)$$

$$z = \sin(\gamma-\alpha)\sin(\beta-\delta)$$

$\Rightarrow$ $$2z = \cos(\gamma-\alpha-\beta+\delta) - \cos(\gamma-\alpha+\beta+\delta)$$

$$2x + 2y + 2z = 0 \quad \Rightarrow \quad x + y + z = 0$$

If $x + y + z = 0$ then $x^3 + y^3 + z^3 = 3xyz$

11. $$X^2 + 4XY + Y^2 = (x\cos\theta - y\sin\theta)^2 + (x\sin\theta + y\cos\theta)^2$$

$$+ 4(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta)$$

$$= x^2 + y^2 + 4\{x^2\sin\theta\cos\theta - y^2\sin\theta\cos\theta + xy(\cos^2\theta - \sin^2\theta)\}$$

$$= x^2(1 + 4\sin\theta\cos\theta) + y^2(1 - 4\sin\theta\cos\theta) + 4xy(\cos^2\theta - \sin^2\theta)$$

$$\cos^2\theta - \sin^2\theta = 0$$

$\Rightarrow$ $$\theta = \frac{\pi}{4} \qquad (0 \le \theta \le \pi/2)$$

$$x^2 + 4XY + Y^2 = 3x^2 - y^2$$

$\Rightarrow$ $A = 3$ and $B = -1$

12. (A) $$2(a+d) = 2(b+c)$$

(B) $$\tan 50^\circ = \frac{1}{\tan 40^\circ} = \frac{1-\tan^2 20^\circ}{2\tan 20^\circ}$$

$\Rightarrow$ $$\tan 20^\circ + 2\tan 50^\circ = \tan 70^\circ$$

$\Rightarrow \qquad 2a + 2b = 2c$

(D) $\quad \tan 20° - 2\tan 10° = \tan 20° \tan^2 10° > 0$

$\Rightarrow \qquad \tan 20° > 2\tan 10°$

$\Rightarrow \qquad b > a$ and $d > c$

13. (A) $\frac{1}{2}(2\sin 75° \cos 75°) = \frac{1}{2}\sin 150° = \frac{1}{4}$

(B) $\log_2^{28} = 2 + \log_2^7$ (irrational)

(C) $\log_3^5 \cdot \log_5^6 = \log_3^6 = 1 + \log_3^2$ (irrational)

(D) $8^{-\log_{27}^{3}} = 8^{-1/3} = \frac{1}{2}$

14. $\alpha - \beta = \sin x \cos x(\cos^2 x - \sin^2 x) = \frac{1}{2}\sin 2x \cos 2x = \frac{1}{4}\sin 4x$

$\alpha + \beta = \sin x \cos x = \frac{1}{2}\sin 2x$

15.

$$\begin{aligned}\sqrt{2 + \sqrt{2 + 2\cos 4\theta}} &= \sqrt{2 + \sqrt{2(1 + \cos 4\theta)}} \\ &= \sqrt{2 + \sqrt{4\cos^2 2\theta}} \\ &= \sqrt{2 + 2|\cos 2\theta|}\end{aligned}$$

If $\pi < 2\theta < 3\pi/2$ then $\frac{\pi}{2} < \theta < \frac{3\pi}{4}$

$$\sqrt{2 + 2|\cos 2\theta|} = \sqrt{2 - 2\cos 2\theta} = 2|\sin\theta| = 2\sin\theta$$

If $\frac{3\pi}{2} < 2\theta < 2\pi$ then $\frac{3\pi}{4} < \theta < \pi$

$$\sqrt{2 + 2|\cos 2\theta|} = \sqrt{2 + 2\cos 2\theta} = 2|\cos\theta| = -2\cos\theta$$

16. $1 + \tan\alpha + \tan^2\alpha = \tan^3\alpha$

$\Rightarrow \quad 1 + \tan^2\alpha = \tan\alpha(\tan^2\alpha - 1)$

18. $\alpha > \frac{1}{\sin^6 x + \cos^6 x} \quad \Rightarrow \alpha > \frac{1}{1 - 3\sin^2 x \cos^2 x}; \quad 1 \le \frac{1}{1 - 3\sin^2 x \cos^2 x} \le 4$

19. $\log_{10} \sin x + \log_{10} \cos x + 2\log_{10} \cot x + \log_{10} \tan x = -1$

$\log_{10}(\sin x \cdot \cos x \cdot \cot x) = k = \log_{10} \cos^2 x = -1$

20. $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$\frac{3}{\tan C} + \frac{6}{\tan C} + \tan C = \frac{3}{\tan C} \cdot \frac{6}{\tan C} \cdot \tan C$

$\Rightarrow \qquad \tan^2 C = 9 \Rightarrow \tan C = 3$

21. $\dfrac{(1-\cot x)}{\sin^2 x} = (1-\cot x)\cdot \operatorname{cosec}^2 x$

$$= (1-\cot x)(1+\cot^2 x)$$

22. $f(x) = \dfrac{1}{2}\left[2\sin^2 x + 2\sin^2\left(x+\dfrac{2\pi}{3}\right) + 2\sin^2\left(x+\dfrac{4\pi}{3}\right)\right]$

23. $y = \dfrac{\tan x}{\tan 3x} = \dfrac{1-3\tan^2 x}{3-\tan^2 x}$

$$\tan^2 x = \frac{1-3y}{3-y} > 0$$

24. $\sqrt{2}\sin(A-B) = \cos B(\sin B - \sin^3 B) - \sin B(\cos B + \cos^3 B)$

$$= -\sin B\cos B$$

$$= -\frac{1}{2}\sin 2B \Rightarrow \sin(A-B) = -\frac{\sin 2B}{2\sqrt{2}}$$

25. $\alpha > \dfrac{1}{\sin^6 x + \cos^6 x} \Rightarrow \alpha > \dfrac{1}{1-3\sin^2 x\cos^2 x}$; $1 \le \dfrac{1}{1-3\sin^2 x\cos^2 x} \le 4$

26. $1+\tan\alpha + \tan^2\alpha = \tan^3\alpha$

$\Rightarrow \quad 1+\tan^2\alpha = \tan\alpha(\tan^2\alpha - 1)$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 3

1. $\theta = 286.5°$ (IV quadrant) $l < 0, m > 0$

2. $\tan(-1042°) = -\tan(1080° - 38°) = \tan 38° < \tan 45°$

3. $\theta = 401.1°$ (I quadrant) $l > 0, m > 0$

Paragraph for Question Nos. 4 to 6

$a = \sin\alpha \quad b = \sin\left(\alpha + \dfrac{2\pi}{3}\right) \quad c = \sin\left(\alpha + \dfrac{4\pi}{3}\right)$

$p = \cos\alpha \quad q = \cos\left(\alpha + \dfrac{2\pi}{3}\right) \quad r = \cos\left(\alpha + \dfrac{4\pi}{3}\right)$

4. $a+b+c=\sin\alpha+\sin\left(\alpha+\frac{2\pi}{3}\right)+\sin\left(\alpha+\frac{4\pi}{3}\right)$

$$=\sin\alpha+2\sin(\alpha+\pi)\cos\left(\frac{\pi}{3}\right)=0$$

5. $ab+bc+ac=\sin\alpha\sin\left(\alpha+\frac{2\pi}{3}\right)+\sin\left(\alpha+\frac{2\pi}{3}\right)\sin\left(\alpha+\frac{4\pi}{3}\right)+\sin\alpha\sin\left(\alpha+\frac{4\pi}{3}\right)$

$$=\frac{1}{2}\left[\cos\frac{2\pi}{3}-\cos\left(2\alpha+\frac{2\pi}{3}\right)+\cos\frac{2\pi}{3}-\cos(2\alpha+2\pi)+\cos\frac{4\pi}{3}-\cos\left(2\alpha+\frac{4\pi}{3}\right)\right]=\frac{-3}{4}$$

6. $qc-rb=\cos\left(\alpha+\frac{2\pi}{3}\right)\sin\left(\alpha+\frac{4\pi}{3}\right)-\cos\left(\alpha+\frac{4\pi}{3}\right)\sin\left(\alpha+\frac{2\pi}{3}\right)=\sin\frac{2\pi}{3}=\frac{\sqrt{3}}{2}$

Paragraph for Question Nos. 7 to 8

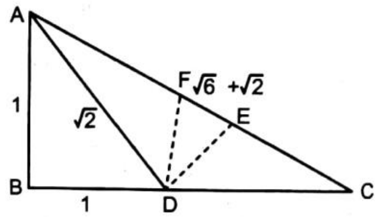
7.

$$\tan A=\sqrt{7+4\sqrt{3}}=\cot C$$

$$\sqrt{\tan A+\cot C}=\sqrt{2\sqrt{7+4\sqrt{3}}}$$

$$=\sqrt{2(2+\sqrt{3})}=\sqrt{4+2\sqrt{3}}$$

$$=\sqrt{3}+1$$



8. $\log_{AE}\left(\frac{AC}{CD}\right)=\log_{\sqrt{2}}\left(\frac{\sqrt{2}+\sqrt{6}}{1+\sqrt{3}}\right)=\log_{\sqrt{2}}^{\sqrt{2}}=1$

Paragraph for Question Nos. 9 to 10

9. In a ΔABC, $\cot A+\cot B+\cot C\geq\sqrt{3}\Rightarrow\cot\theta\geq\sqrt{3}$

10. $\cot\theta-\cot A=\cot B+\cot C\Rightarrow\sin(A-\theta)=\frac{\sin^2 A\sin\theta}{\sin B\sin C}$

$\sin(B-\theta)=\frac{\sin^2 B\cdot\sin\theta}{\sin A\sin C}$ and $\sin(C-\theta)=\frac{\sin^2 C\sin\theta}{\sin A\sin B}$

Paragraph for Question Nos. 11 to 12

11. $f(x)=\dfrac{\left|\cos\frac{x}{2}\right|+\left|\sin\frac{x}{2}\right|}{\left|\cos\frac{x}{2}\right|-\left|\sin\frac{x}{2}\right|}$

12. If $\frac{\pi}{2}<\frac{x}{2}<\pi\Rightarrow f(x)=\dfrac{-\cos\frac{x}{2}+\sin\frac{x}{2}}{-\cos\frac{x}{2}-\sin\frac{x}{2}}=\dfrac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}$

Exercise-4 : Matching Type Problems

1. (A) If $A+B=45°$ then $(1+\tan A)(1+\tan B)=2$

(B) $a^2-5a\le 6\sin x \;\; \forall \;\; x\in R$

$a^2-5a\le -6$

$a^2-5a+6\le 0 \quad \Rightarrow \quad (a-3)(a-2)\le 0$

(C) $$\frac{\left(a+\frac{1}{a}\right)^4-\left(a^4+\frac{1}{a^4}+2\right)}{\left(a+\frac{1}{a}\right)^2+a^2+\frac{1}{a^2}}=\frac{\left(a+\frac{1}{a}\right)^4-\left(a^2+\frac{1}{a^2}\right)^2}{\left(a+\frac{1}{a}\right)^2+a^2+\frac{1}{a^2}}=\left(a+\frac{1}{a}\right)^2-\left(a^2+\frac{1}{a^2}\right)=2$$

(D) $\sum_{k=1}^{3}(x-k)^2=(x-1)^2+(x-2)^2+(x-3)^2=0$ No real root

2. (A) $y=\frac{1-\tan^2(\pi/4-x)}{1+\tan^2(\pi/4-x)}=\cos(\pi/2-2x)=\sin 2x$

(B) $0\le \log_3\left(\frac{5\sin x-12\cos x+26}{13}\right)\le 1$

(C) $y=-2\sin^2 x+\cos x+3=2\cos^2 x+\cos x+1=2\left(\cos x+\frac{1}{4}\right)^2+\frac{7}{8}$

(D) $y=4\sin^2\theta+4\sin\theta\cos\theta+\cos^2\theta=(2\sin\theta+\cos\theta)^2$

4. (A) $\cos^2 x=\left(\frac{1}{5}-\sin x\right)^2$

$\Rightarrow$ $(5\sin x-4)(5\sin x+3)=0$

$\Rightarrow$ $\sin x=\frac{4}{5}$ or $-\frac{3}{5}$

(B) $\cot\frac{\theta}{2}=1+\cot\theta$

$\Rightarrow$ $2\cos^2\frac{\theta}{2}=\cos\theta+\sin\theta$

$\Rightarrow$ $\sin\theta=1 \quad \Rightarrow \quad \theta=-\frac{3\pi}{2},\frac{\pi}{2}$

(C) $f(x)=-\sin^4 x+8\sin^2 x+2$

$\Rightarrow$ $f(x)\in[2,9]$

(D) $\log_2\frac{(2x^2+5x+27)}{(2x-1)^2}\ge 0 \qquad \left(x>\frac{1}{2}\right)$

$$\Rightarrow \qquad 2x^2 - 9x - 26 \le 0$$

$$\Rightarrow \qquad -2 \le x \le \frac{13}{2}$$

5. (A) $f(x) = -2\sin^2 x + \sin x - 6$

$y_{\min} = -9$ at $\sin x = -1$

$y_{\max} = -\frac{47}{8}$ at $\sin x = \frac{1}{4}$

(B) $f(x) = 2\cos^2 x + 6$

$y_{\min} = 6;\ y_{\max} = 8$

(C) $f(x) = \frac{1}{2}[4\sin 2x - 1 + \cos 2x + 3(1 + \cos 2x)]$

$= \frac{1}{2}[2 + 4\sin 2x + 4\cos 2x]$

$= 1 + 2(\sin 2x + \cos 2x)$

$y_{\max} = 1 + 2\sqrt{2};\ y_{\min} = 1 - 2\sqrt{2}$

(D) $f(x) = \sqrt{2}\sin\left(\frac{\pi}{4} + \sin x\right)$

Exercise-5 : Subjective Type Problems

1. $\dfrac{\sin 80° \sin 65° \sin 35°}{2\sin 35° \cos 15° + 2\sin 35° \cos 35°} = \dfrac{\sin 80° \sin 65°}{2(\cos 15° + \cos 35°)} = \dfrac{\sin 80° \sin 65°}{4\cos 25° \cos 10°} = \dfrac{1}{4}$

2. If $A + B = 45°$

$$(1 - \cot A)(1 - \cot B) = 2$$

$$\Rightarrow \quad (1 - \cot 23°)(1 - \cot 22°) = 2$$

3.
$$4x^2 - 7x + 1 = 0$$

$$\tan A + \tan B = \frac{7}{4}$$

$$\tan A \cdot \tan B = \frac{1}{4}$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A + \tan B} = \frac{7}{3}$$

$$4\sin^2(A + B) - 7\sin(A + B)\cos(A + B) + \cos^2(A + B)$$

$$= \frac{4\tan^2(A + B) - 7\tan(A + B) + 1}{1 + \tan^2(A + B)} = 1$$

4. $\dfrac{(18-2)\times 180^\circ}{18}+\dfrac{(n-2)\times 180^\circ}{n}+60^\circ=360^\circ \Rightarrow n=9$

5. $10(1-\cos 2\alpha)^2+15(1+\cos 2\alpha)^2=24$

$$\Rightarrow \quad (5\cos 2\alpha+1)^2=0 \quad \Rightarrow \quad \cos 2\alpha=-\frac{1}{5} \quad \Rightarrow \quad \tan^2\alpha=\frac{3}{2}$$

6. $\tan\left(\dfrac{3\pi}{8}-\dfrac{\pi}{8}\right)\left(\tan\dfrac{3\pi}{8}-\tan\dfrac{\pi}{8}\right)+\tan\left(\dfrac{5\pi}{8}-\dfrac{3\pi}{8}\right)\left(\tan\dfrac{5\pi}{8}-\tan\dfrac{3\pi}{8}\right)$

$$+\tan\left(\frac{7\pi}{8}-\frac{5\pi}{8}\right)\left(\tan\frac{7\pi}{8}-\tan\frac{3\pi}{8}\right)+\tan\left(\frac{9\pi}{8}-\frac{7\pi}{8}\right)\left(\tan\frac{9\pi}{8}-\tan\frac{7\pi}{8}\right)=\tan\frac{9\pi}{8}-\tan\frac{\pi}{8}=0$$

7. $\dfrac{\cos\dfrac{2\pi}{7}+2\cos^2\dfrac{\pi}{7}}{\cos\dfrac{\pi}{7}\cos\dfrac{2\pi}{7}}=\dfrac{4\left(\cos\dfrac{2\pi}{7}+2\cos^2\dfrac{\pi}{7}\right)\sin\dfrac{\pi}{7}}{\sin\dfrac{4\pi}{7}}=\dfrac{4\left(1+2\cos\dfrac{2\pi}{7}\right)\sin\dfrac{\pi}{7}}{\sin\dfrac{3\pi}{7}}$

$$=\frac{4\left(1+2\cos\dfrac{2\pi}{7}\right)\sin\dfrac{\pi}{7}}{\sin\dfrac{\pi}{7}\left(3-4\sin^2\dfrac{\pi}{7}\right)}=4$$

8.
$$a^2\sec^2 200^\circ=c^2\tan^2 200^\circ+d^2+2cd\tan 200^\circ$$
$$b^2\sec^2 200^\circ=c^2+d^2\tan^2 200^\circ-2cd\tan 200^\circ$$
$$\Rightarrow \quad a^2+b^2=c^2+d^2$$
$$(a\sec 200^\circ-c\tan 200^\circ)^2=d^2$$
$$(b\sec 200^\circ+d\tan 200^\circ)^2=c^2$$
$$\Rightarrow \quad (c^2+d^2)(\sec^2 200^\circ+\tan^2 200^\circ)+(2bd-2ac)\sec 200^\circ\tan 200^\circ=c^2+d^2$$
$$\Rightarrow \quad (c^2+d^2)(2\tan^2 200^\circ)=(2ac-2bd)\sec 200^\circ\tan 200^\circ$$
$$\Rightarrow \quad \frac{2(c^2+d^2)}{ac-bd}=\frac{2\sec 200^\circ}{\tan 200^\circ}=\frac{2}{\sin 200^\circ}=\frac{-2}{\sin 20^\circ}$$

9. $2\cos\dfrac{\pi}{17}\cos\dfrac{9\pi}{17}+\cos\dfrac{7\pi}{17}+\cos\dfrac{9\pi}{17}=\cos\dfrac{10\pi}{17}+\cos\dfrac{8\pi}{17}+\cos\dfrac{7\pi}{17}+\cos\dfrac{9\pi}{17}=0$

10. $\dfrac{\cos\theta-\cos 3\theta+\cos 3\theta-\cos 9\theta+\cos 9\theta-\cos 17\theta}{\sin 3\theta-\sin\theta+\sin 9\theta-\sin 3\theta+\sin 17\theta-\sin 9\theta}=\dfrac{\cos\theta-\cos 17\theta}{\sin 17\theta-\sin\theta}=\tan 9\theta$

11. $8abc=8\sin 10^\circ\sin 50^\circ\sin 70^\circ=1$

$$\frac{a+b}{c}=\frac{\sin 10^\circ+\sin 50^\circ}{\sin 70^\circ}=\frac{2\sin 30^\circ\cos 20^\circ}{\sin 70^\circ}=1$$

$$\frac{1}{a}+\frac{1}{b}-\frac{1}{c}=\frac{1}{\sin 10^\circ}+\frac{1}{\sin 50^\circ}-\frac{1}{\sin 70^\circ}=\frac{\sin 50^\circ\sin 70^\circ+\sin 10^\circ\sin 70^\circ-\sin 10^\circ\sin 50^\circ}{\sin 10^\circ\sin 50^\circ\sin 70^\circ}=6$$

12. $\frac{1}{4}\left[4\sin^3\theta+4\sin^3\left(\theta+\frac{2\pi}{3}\right)+4\sin^3\left(\theta+\frac{4\pi}{3}\right)\right]$

$$=\frac{1}{4}\left[3\sin\theta-\sin3\theta+3\sin\left\{\left(\theta+\frac{2\pi}{3}\right)-\sin(3\theta+2\pi)+3\sin\left(\theta+\frac{4\pi}{3}\right)-\sin(3\theta+4\pi)\right\}\right]$$

$$=\frac{1}{4}\left[3\left\{\sin\theta+\sin\left(\theta+\frac{2\pi}{3}\right)+\sin\left(\theta+\frac{4\pi}{3}\right)\right\}-3\sin3\theta\right]=-\frac{3}{4}\sin3\theta$$

13. $\sum_{r=1}^{n}\frac{\sin(2^r-2^{r-1})}{\cos2^r\cos2^{r-1}}=\sum_{r=1}^{n}(\tan2^r-\tan2^{r-1})=\tan2^n-\tan1$

14. $x=\sec\theta-\tan\theta,\ y=\operatorname{cosec}\theta+\cot\theta$

$$y-x-xy=\frac{1+\cos\theta}{\sin\theta}-\frac{1-\sin\theta}{\cos\theta}-\frac{(1-\sin\theta)(1+\cos\theta)}{\sin\theta\cdot\cos\theta}=1$$

15.
$$\cos18^\circ-\cos72^\circ=2\sin45^\circ\sin27^\circ$$
$$=\sqrt{2}\sin27^\circ$$

16. $3(\sin1-\cos1)^4+6(\sin1+\cos1)^2+4(\sin^6 1+\cos^6 1)$

$$=3(1-2\sin1\cos1)^2+6(1+2\sin1\cos1)+4(1-3\sin^2 1\cos^2 1)$$
$$=3(1+4\sin^2 1\cos^2 1-4\sin1\cos1)+10+12\sin1\cos1-12\sin^2 1\cos^2 1$$
$$=13$$

17. $3^{\sin2x+2\cos^2x}+\frac{3^3}{3^{\sin2x+2\cos^2x}}=28$

Let $3^{\sin2x+2\cos^2x}=t,\quad t^2-28t+27=0\Rightarrow t=1,27$

If $t=1\Rightarrow\ \sin2x+2\cos^2x=0$

$$2\cos x(\sin x+\cos x)=0\quad\Rightarrow\quad x=\frac{\pi}{2},\frac{3\pi}{4},\frac{7\pi}{4}$$

If $t=27$

$\Rightarrow\ \sin2x+2\cos^2x=3$ (Not possible)

$(\sin2\alpha-\cos2\alpha)^2+8\sin4\alpha=1+7\sin4\alpha=1$ (at $\alpha=\frac{\pi}{2},\frac{3\pi}{4},\frac{7\pi}{4}$)

18. $(\sin\theta+\operatorname{cosec}\theta)^2+(\cos\theta+\sec\theta)^2$

$$=5+\operatorname{cosec}^2\theta+\sec^2\theta$$
$$=7+\tan^2\theta+\cot^2\theta$$
$$\geq9$$

19. $\tan20^\circ+\tan40^\circ+\tan80^\circ-\tan60^\circ$

$$=\frac{\sin20^\circ\cos80^\circ+\sin80^\circ\cos20^\circ}{\cos20^\circ\cos80^\circ}+\frac{\sin40^\circ\cos60^\circ-\sin60^\circ\cos40^\circ}{\cos40^\circ\cos60^\circ}$$

$$= \frac{\sin 100^\circ}{\cos 20^\circ \cos 80^\circ} - \frac{\sin 20^\circ}{\cos 40^\circ \cos 60^\circ}$$

$$= \frac{\sin 80^\circ}{\cos 20^\circ \cos 80^\circ} - \frac{2\sin 20^\circ}{\cos 40^\circ}$$

$$= \frac{\sin 80^\circ \cos 40^\circ - \sin 40^\circ \cos 80^\circ}{\cos 20^\circ \cos 40^\circ \cos 80^\circ} = \frac{\sin 40^\circ}{\cos 20^\circ \cos 40^\circ \cos 80^\circ}$$

$$= \frac{8\sin 40^\circ \sin 20^\circ}{\sin(8\times 20^\circ)} = 8\sin 40^\circ$$

20. $1 + \cos 10x \cos 6x = 2\cos^2 8x + \sin^2 8x$

$2 + \cos 16x + \cos 4x = 2(1 + \cos 16x) + 1 - \cos 16x$

$\Rightarrow$ $\cos 4x = 1$

$x = \frac{n\pi}{2}$ $(n = 0, \pm 1, \pm 2, \pm 3 \ldots\ldots)$

If $360^\circ < k < 540^\circ$

$\Rightarrow$ $k = 450^\circ$ $(n = 5)$

21. $\cos 20^\circ + 2\sin^2 55^\circ = 1 + \sqrt{2}\sin k^\circ$

$= \cos 20^\circ + 1 - \cos 110^\circ$

$= 1 + \cos 20^\circ + \sin 20^\circ$

$= 1 + \sqrt{2}\sin(45^\circ + 20^\circ)$

$\Rightarrow$ $k = 65$

23. $\tan 19x = \frac{\cos 96^\circ + \cos 6^\circ}{\cos 96^\circ - \cos 6^\circ} = -\frac{2\cos 51^\circ \cos 45^\circ}{2\sin 51^\circ \sin 45^\circ} = -\cot 51^\circ = \tan 141^\circ$

$\Rightarrow 19x = 180^\circ n + 141$

24. $\frac{2\sin 40^\circ + \sin 20^\circ}{\cos 20^\circ \cos 30^\circ} = \frac{2\sin(60^\circ - 20^\circ) + \sin 20^\circ}{\cos 20^\circ \cos 30^\circ}$

25. $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} - 1 = \frac{\sin\frac{3\pi}{7}}{\sin\frac{\pi}{7}} \cdot \cos\frac{4\pi}{7} - 1 = \frac{-3}{2}$

26. $\frac{k}{2}(\cos 2A - \cos 3A) = \frac{11}{8}$

$\frac{k}{2}[2\cos^2 A - 1 - 4\cos^3 A + 3\cos A] = \frac{11}{8}$

$\Rightarrow$ $k = 4$

27. $3\sin^2 x + 4\cos^2 x = 3 + \cos^2 x$

28. $\tan\alpha + \tan\beta = 12$

$\tan\alpha \cdot \tan\beta = -3$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=3$$

29. $$\frac{\cos 24^\circ\cos 33^\circ}{2\sin 33^\circ\sin^2 57^\circ}+\left(\frac{\sin 18^\circ\cos 9^\circ}{\sin 9^\circ}-\cos 18^\circ\right)$$

$$\frac{\cos 24^\circ\cos 33^\circ}{\sin 57^\circ\cos 24^\circ}+\frac{\sin 9^\circ}{\sin 9^\circ}=2$$

30. $$\tan\theta\left(\frac{1+\cos 2\theta}{\cos 2\theta}\right)\left(\frac{1+\cos 4\theta}{\cos 4\theta}\right)\left(\frac{1+\cos 8\theta}{\cos 8\theta}\right)$$

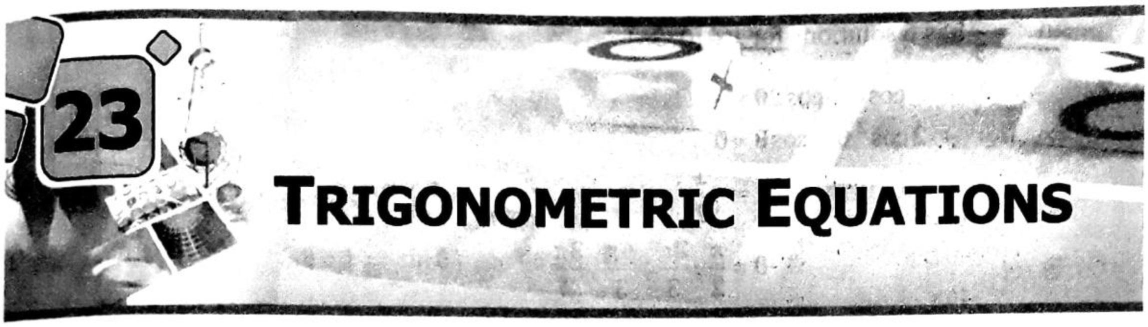
$$\frac{\sin\theta}{\cos\theta}\left(\frac{2\cos^2\theta}{\cos 2\theta}\right)\left(\frac{2\cos^2 2\theta}{\cos 4\theta}\right)\left(\frac{2\cos^2 4\theta}{\cos 8\theta}\right)=\frac{8\sin\theta\cos\theta\cos 2\theta\cos 4\theta}{\cos 8\theta}=\frac{\sin 8\theta}{\cos 8\theta}=\tan 8\theta$$

31. $y=\sin^2 x+\cos^2 x+\tan^2 x+\cot^2 x+\operatorname{cosec}^2 x+\sec^2 x+6$

$y=9+2(\tan^2 x+\cot^2 x)\ge 13$

* * *

Chapter 23 – Trigonometric Equations



23

TRIGONOMETRIC EQUATIONS

Exercise-1 : Single Choice Problems

1. $\tan^2 x - \sec^2 y = \frac{5a}{6} - 3 = -2 - a^2 \quad \Rightarrow 6a^2 + 5a - 6 = 0$

2. $[\tan(x+y) - \cot(x+y)]^2 + (x+1)^2 = 0$

$\Rightarrow \quad x = -1 \quad \text{and} \quad \tan^2(x+y) = 1$

$$x + y = n\pi \pm \frac{\pi}{4}$$

3.

$$\sin x + \cos x = 1$$

$$\sin\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$x = n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{4}$$

4. $\sin^2(\sin x) - 3\sin(\sin x) + 2 = 0$

$\{\sin(\sin x) - 2\}\{\sin(\sin x) - 1\} = 0$

Equation has no solution.

5. $\tan 2x = \tan 6x \quad \Rightarrow \quad \sin 4x = 0$

$$4x = \pi, 2\pi, 3\pi, \ldots\ldots, 11\pi$$

$$x = \frac{\pi}{4}, \frac{2\pi}{4}, \frac{3\pi}{4}, \ldots\ldots, \frac{11\pi}{4}$$

But $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$ are rejected. So number of solutions = 5.

6. $3\sin^2 x - 6\sin x - \sin x + 2 = 0$

$(3\sin x - 1)(\sin x - 2) = 0$

$\sin x \neq 2$, then

$$\sin x = \frac{1}{3}$$

$\sin x = \frac{1}{3}$ has 6 solutions for $x \in [0, 5\pi]$

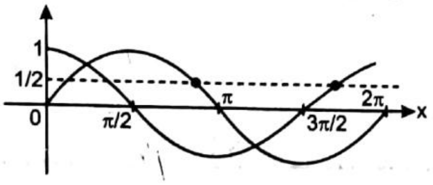
7. $$\cos\theta + \cos 2\theta = -1$$

$\Rightarrow \quad 2\cos^2\theta + \cos\theta = 0$

$\Rightarrow \quad \cos\theta = 0$ or $\cos\theta = -\frac{1}{2}$

$\Rightarrow \quad \theta = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2}$

8.



in $[0, 2\pi)$ max. $(\sin x, \cos x) = \frac{1}{2}$ has two solutions.

9. $(\cot^2 x + 2\sqrt{3}\cot x + 3) + (\cot^2 x + 1) + (4\operatorname{cosec} x + 4) = 0$

$(\cot x + \sqrt{3})^2 + (\operatorname{cosec} x + 2)^2 = 0$

$\Rightarrow \quad \cot x = -\sqrt{3}$ and $\operatorname{cosec} x = -2$

10. $$\sin^2 x = \sin^2 3x$$

$\Rightarrow \quad 3x = n\pi \pm x$

$x = \frac{n\pi}{4}$, $x = \frac{n\pi}{2}$, hence general solution is $\frac{n\pi}{4}$.

11. $\sin x > 0$

$\Rightarrow \quad 8\sin^2 x\cos^2 x = 1$

$\Rightarrow \quad 2\sin^2 2x = 1$

$\Rightarrow \quad \cos 4x = 0$

$$x = (2n+1)\frac{\pi}{8} \quad (n \in I)$$

12. $$\cos x + \cos 2x + \cos 3x + \cos 4x + \cos 5x = 5$$

$\Rightarrow \quad \cos x = 1 \cap \cos 2x = 1 \cap \cos 3x = 1 \cap \cos 4x = 1 \cap \cos 5x = 1$

$x = 2n\pi \cap x = n\pi \cap x = \frac{2n\pi}{3} \cap x = \frac{2n\pi}{4} \cap x = \frac{2n\pi}{5}$

$\Rightarrow \quad x = 2n\pi$

13. $(2\sin x - \operatorname{cosec} x)^2 + (\tan x - \cot x)^2 = 0$

$\Rightarrow \quad \sin^2 x = \frac{1}{2} \cap \tan^2 x = 1 \Rightarrow x = n\pi \pm \frac{\pi}{4}$

14.
$$\cos^3 3x + \cos^3 5x = (2\cos 4x \cos x)^3$$
$$\cos^3 3x + \cos^3 5x = (\cos 5x + \cos 3x)^3$$
$$\Rightarrow \quad 3\cos 5x \cos 3x(\cos 5x + \cos 3x) = 0$$
$$\Rightarrow \quad \cos 5x \cos 3x \cdot \cos 4x \cos x = 0$$

15. $\sin^{100} x = 1 + \cos^{100} x \Rightarrow \sin^{100} x = 1$ and $\cos^{300} x = 0$

16. $\sin\theta \le 1$ and $\sec^2 4\theta \ge 1 \Rightarrow \sin\theta = \sec 4\theta = 1;\ \theta = \dfrac{\pi}{2}$

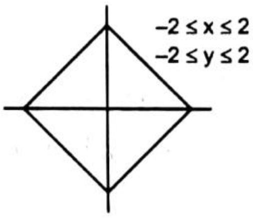
17. $(4\sin^2 x + \operatorname{cosec}^2 x) + (\tan^2 x + \cot^2 x) = 6$
$$(2\sin x - \operatorname{cosec} x)^2 + (\tan x - \cot x)^2 = 0$$
$$\Rightarrow \quad 2\sin x = \operatorname{cosec} x \quad \text{and} \quad \tan x = \cot x$$

18. $\sin^4\theta - 2\sin^2\theta + 1 = 2$

$(\sin^2\theta - 1)^2 = 2 = \cos^4\theta$ (not possible)

19. $\cos(P\sin x) = \sin(P\cos x)$
$$\cos(P\sin x) = \cos\left(\frac{\pi}{2} - P\cos x\right)$$
$$P\sin x + P\cos x = 2n\pi + \frac{\pi}{2}$$
$$P\sin x - P\cos x = 2n\pi - \frac{\pi}{2}$$

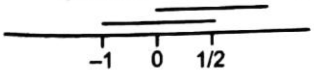
20. $|x| + |y| = 2$



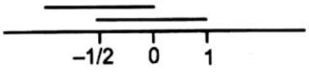
$$\sin\left(\frac{\pi x^2}{3}\right) = 1$$
$$\frac{\pi x^2}{3} = \frac{\pi}{2}$$
$$x = \pm\sqrt{\frac{3}{2}}$$

21. $x \in \left(-\dfrac{\pi}{2}, \pi\right)$
$$\cos 2x > |\sin x|$$

$\sin x \ge 0$
$$1 - 2\sin^2 x - \sin x > 0$$
$$2\sin^2 x + 2\sin x - \sin x - 1 < 0$$
$$(2\sin x - 1)(\sin x + 1) < 0$$



$\sin x < 0$
$$2\sin^2 x - 2\sin x + \sin x - 1 < 0$$
$$(2\sin x + 1)(\sin x - 1) < 0$$



$$-\frac{1}{2} < \sin x \le \frac{1}{2}$$

$$\left(-\frac{\pi}{6}, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, \pi\right)$$

22. $\sin^4 x + \cos^4 x = \sin x \cos x$

$$1 - 2\sin^2 x \cos^1 x = \sin x \cos x$$

$$2y^2 + y - 1 = 0$$

$$(2y-1)(y+1) = 0$$

$$y = \frac{1}{2} \qquad\qquad y = -1$$

$$2\sin x \cos x = 1 \qquad\qquad \sin x \cos x \ne -1$$

$$\sin 2x = 1$$

$$2x = \frac{\pi}{2}, \frac{5\pi}{2}$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$

23. $\sin\frac{5x}{2} = 1 \cap \sin\frac{x}{2} = -1$

24. $\cos 2\theta = \sin^2\theta \Rightarrow \sin^2\theta = \frac{1}{3} \Rightarrow \sin\theta = \pm\frac{1}{\sqrt{3}}$

25. $b\sin\theta = -c - a\cos\theta$

$$b^2(1-\cos 2\theta) = c^2 + a^2\cos 2\theta - 2ac\cos\theta$$

$$\Rightarrow \quad (a^2+b^2)\cos 2\theta - 2ac\cos\theta + (c^2-b^2) = 0$$

$$\cos\alpha \cdot \cos\beta = \frac{c^2-b^2}{a^2+b^2} \qquad \ldots(1)$$

$$a^2(1-\sin 2\theta) = c^2 + b^2\sin 2\theta - 2bc\sin\theta$$

$$(a^2+b^2)\sin 2\theta - 2bc\sin\theta + (c^2-a^2) = 0$$

$$\sin\alpha \cdot \sin\beta = \frac{c^2-a^2}{a^2+b^2} \qquad \ldots(2)$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \frac{a^2-b^2}{a^2+b^2}$$

Exercise-2 : One or More than One Answer is/are Correct

1. $2\cos^2\theta + 2\sqrt{2}\cos\theta - 3 = 0$

$$(\sqrt{2}\cos\theta + 1)^2 = 4 \Rightarrow \cos\theta = \frac{1}{\sqrt{2}} \text{ or } \frac{-3}{\sqrt{2}} \text{ (Not possible)}$$

3. $$4\sin 3x + 5 \geq 4\cos 2x + 5\sin x$$

$\Rightarrow$ $(\sin x - 1)(4\sin x + 1)^2 \leq 0 \ \forall\ x \in R$

4. $4\cos x(2 - 3\sin^2 x) + \cos 2x + 1 = 0$

$\cos x(3\cos x + 2)(2\cos x - 1) = 0$

Least difference $= \frac{\pi}{6}$

5. $\cos x \cos 6x = -1$

Case-1 : $\cos x = 1$ and $\cos 6x = -1$

Not possible

Case-2 : $\cos 6x = 1$ and $\cos x = -1$

$\Rightarrow$ $x = (2n-1)\pi, \ (n \in I)$

7. $2k = \sin^2 2x - 2\sin 2x - 2$

Let $\sin 2x = t$ $\quad t \in [-1, 1]$

$$2k = t^2 - 2t - 2 \Rightarrow k \in \left[-\frac{3}{2}, \frac{1}{2}\right]$$

8. $f(\theta) = \left(\cos\theta - \cos\frac{\pi}{8}\right)\left(\cos\theta - \cos\frac{3\pi}{8}\right)\left(\cos\theta - \cos\frac{5\pi}{8}\right)\left(\cos\theta - \cos\frac{7\pi}{8}\right) = \cos^4\theta - \cos^2\theta + \frac{1}{8}$

9. $\dfrac{4\sin^2 x\cos^2 x + 4\sin^4 x - 4\sin^2 x\cos^2 x}{4\cos^2 x - 4\sin^2 x\cos^2 x} = \tan^4 x = \frac{1}{9}$

$\Rightarrow$ $\tan x = \pm\frac{1}{\sqrt{3}}$

10. $\tan\theta(1 - \sin^2\theta) + \cot\theta(1 - \cos^2\theta) + 1 + \sin 2\theta = 0 \Rightarrow \sin 2\theta = -\frac{1}{2}$

11. $2\left(\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta\right) = -(x-3)^2 - 2$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 3

1. $h(x) = f^2(x) + g^2(x) = 2 + 2\sin 4x$

$$\Rightarrow \quad 8\cos 4x \geq 0$$

$$\Rightarrow \quad \cos 4x \geq 0$$

Longest interval $= \dfrac{\pi}{4}$

2. $2 + 2\sin 4x = 4$

$\Rightarrow \quad \sin 4x = 1$

$$\Rightarrow \quad x = (4n+1)\frac{\pi}{8}$$

3.

$$\sin 3x + \cos x = \cos 3x + \sin x$$

$$\Rightarrow \quad \sin 3x - \sin x = \cos 3x - \cos x$$

$$\Rightarrow \quad 2\sin x \cos 2x = -2\sin 2x \sin x$$

$$\Rightarrow \quad \sin x = 0 \quad \text{or} \quad \tan 2x = -1$$

$$\Rightarrow \quad x = 0, \pi \quad \text{or} \quad x = \frac{3\pi}{8}, \frac{7\pi}{8}$$

Exercise-4 : Matching Type Problems

1. (A)

$$\cos^2 x = \left(\frac{1}{5} - \sin x\right)^2$$

$$\Rightarrow \quad (5\sin x - 4)(5\sin x + 3) = 0$$

$$\Rightarrow \quad \sin x = \frac{4}{5} \quad \text{or} \quad -\frac{3}{5}$$

(B)

$$\cot\frac{\theta}{2} = 1 + \cot\theta$$

$$\Rightarrow \quad 2\cos^2\frac{\theta}{2} = \cos\theta + \sin\theta$$

$$\Rightarrow \quad \sin\theta = 1 \ \Rightarrow \ \theta = -\frac{3\pi}{2}, \frac{\pi}{2}$$

(C)

$$f(x) = -\sin^4 x + 8\sin^2 x + 2$$

$$\Rightarrow \quad f(x) \in [2, 9]$$

(D)

$$\log_2 \frac{(2x^2 + 5x + 27)}{(2x-1)^2} \geq 0 \qquad \left(x > \frac{1}{2}\right)$$

$$\Rightarrow \quad 2x^2 - 9x - 26 \leq 0$$

$\Rightarrow \quad -2 \le x \le \frac{13}{2}$

2. (A) $\sin x = 1, \cos y = 1$ or $\sin x = -1, \cos y = -1$

(B) $f'(x) = \cos x + \sin x - K$

$\Rightarrow \quad k \ge \sqrt{2}$

(C) $|x^2 - 1| \le 1$ and $|2x^2 - 5| \le 1$

$\Rightarrow \quad x^2 = 2$

(D) $\sin x + \sin y = \sin(x + y)$

$$\Rightarrow \quad 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$

$$\Rightarrow \quad \sin\left(\frac{x+y}{2}\right) = 0 \quad \text{or} \quad \cos\left(\frac{x-y}{2}\right) = \cos\left(\frac{x+y}{2}\right)$$

$$\Rightarrow \quad x + y = 2n\pi \quad \text{or} \quad x = \frac{n\pi}{2},\ y = \frac{n\pi}{2}$$

if $x = 0,\ y = \pm 1$

if $x = \frac{1}{2},\ y = -\frac{1}{2}$

if $x = -\frac{1}{2},\ y = \frac{1}{2}$

if $y = 0,\ x = \pm 1$

Exercise-5 : Subjective Type Problems

1. Let $\sin x - 1 = a,\ \cos x - 1 = b,\ \sin x = c$

$\Rightarrow \quad a^3 + b^3 + c^3 = (a + b + c)^3$

$\Rightarrow \quad a + b = 0$ or $b + c = 0$ or $c + a = 0$

$\sin x + \cos x = 2$ or $\sin x + \cos x = 1$ or $\sin x = \frac{1}{2}$

$\Rightarrow$ Total solution = 5

2. $\sin y - 2014 \cos y = 1$

$\Rightarrow \quad y = \frac{\pi}{2}$

3. $\frac{2\sin 6x}{\sin x - 1} < 0$

$\Rightarrow \quad \sin 6x > 0$

$\Rightarrow \quad x \in \left(0, \frac{\pi}{6}\right) \cup \left(\frac{\pi}{3}, \frac{\pi}{2}\right)$

$$1+\tan^2 x-2\sqrt{2}\tan x\le 0$$

$$\Rightarrow \qquad x\in\left[\frac{\pi}{8},\frac{3\pi}{8}\right]\Rightarrow x\in\left[\frac{\pi}{8},\frac{\pi}{6}\right)\cup\left(\frac{\pi}{8},\frac{3\pi}{8}\right]$$

4. $\sin^4 x-4\sin^2 x+(2+k)=0$

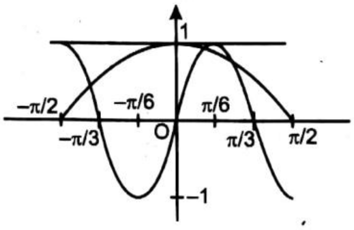
Let $\qquad \sin^2 x=t \qquad t\in[0,1]$

$$t^2-4t+(2+k)=0$$

$$f(0)\,f(1)\le 0$$

$$(k+2)(k-1)\le 0 \;\Rightarrow\; -2\le k\le 1$$

5.



6. $2\sin^2 x+\sin^2 2x=2$

$$2\sin^4 x-3\sin^2 x+1=0\Rightarrow(2\sin^2 x-1)(\sin^2 x-1)=0$$

$$\sin 2x+\cos 2x=\tan x$$

$$2\tan x+1-\tan^2 x=\tan x(1+\tan^2 x)$$

$$\Rightarrow\quad \tan^3 x+\tan^2 x-\tan x-1=0\;\Rightarrow(1+\tan x)(\tan^2 x-1)=0$$

$$2\cos^2 x+\sin x\le 2$$

$$2\sin^2 x-\sin x\ge 0$$

$$\sin x(2\sin x-1)\ge 0$$

7. $(3\cot\theta+1)(\cot\theta+3)=0$

$\cot\theta=-\dfrac{1}{3}$ and $\cot\theta=-3$

$\theta=\alpha,\pi+\alpha \qquad \theta=\dfrac{\pi}{2}-\alpha,\pi+\dfrac{\pi}{2}-\alpha$

8. $(8\cos 4\theta-3)(\cot\theta-\tan\theta)^2=12$

$$8(2\cos^2 2\theta-1)-3\left(\frac{4\cos^2 2\theta}{\sin^2 2\theta}\right)=12$$

$$16\cos^4 2\theta-8\cos^2 2\theta-3=0$$

$$\Rightarrow\quad(4\cos^2 2\theta-3)(4\cos^2 2\theta+1)=0$$

$$\Rightarrow \qquad \cos 2\theta=\pm\frac{\sqrt{3}}{2}$$

9. $2\sin^2 x + 4\sin^2 x\cos^2 x = 2$

$2\sin^4 x - 3\sin^2 x + 1 = 0 \quad \Rightarrow (\sin^2 x - 1)(2\sin^2 x - 1) = 0$

$$\sin x = \pm\frac{1}{\sqrt{2}}, \pm 1$$

$$\sin 2x + \cos 2x = \tan x$$

$$\frac{2\tan x}{1+\tan^2 x} + \frac{1-\tan^2 x}{1+\tan^2 x} = \tan x$$

$$\Rightarrow \quad \tan^3 x + \tan^2 x - \tan x - 1 = 0$$

$$(\tan x + 1)^2(\tan x - 1) = 0$$

$$2\cos^2 x + \sin x \le 2$$

$$2\sin^2 x - \sin x \ge 0$$

$$\sin x(2\sin x - 1) \ge 0$$

* * *

Chapter 24 – Solution of Triangles

24

SOLUTION OF TRIANGLES

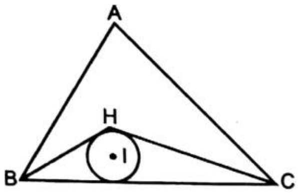
Exercise-1 : Single Choice Problems

1. $$\frac{\cot A+\cot B}{\cot C}=\frac{\cos A\sin B+\cos B\sin A}{(\sin A\sin B)\dfrac{\cos C}{\sin C}}=\frac{\sin^2 C}{(\sin A\sin B)\cos C}=\frac{c^2}{ab\cdot\dfrac{a^2+b^2-c^2}{2ab}}$$

$$=\frac{2c^2}{\left(\dfrac{17}{9}-1\right)c^2}=\frac{18}{8}$$

2. $$\angle BIC=\frac{\pi}{2}+\left(\frac{\pi-A}{2}\right)$$

$$=\frac{\pi}{2}+\left(\frac{B+C}{2}\right)$$



3. $$\frac{1}{64}[(2R\cos A)^2+a^2][(2R\cos B)^2+b^2][(2R\cos C)^2+c^2]$$

$$\frac{1}{64}[(2R\cos A)^2+(2R\sin A)^2][(2R\cos B)^2+(2R\sin B)^2][(2R\cos C)^2+(2R\sin C)^2]$$

$$=R^6$$

4. $B=60°$

$$2\sin^2 B=3\sin^2 C\Rightarrow\sin C=\frac{1}{\sqrt{2}}\Rightarrow C=45°$$

5. $$\tan\frac{A}{2}\tan\frac{C}{2}=\frac{1}{3}$$

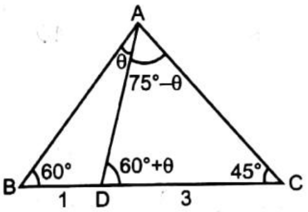
$$\frac{s-b}{s}=\frac{1}{3}\Rightarrow b=\frac{2}{3}s\Rightarrow\frac{a+c}{2}=b\Rightarrow b\geq 2\qquad(\text{A.M.}\geq\text{G.M.})$$

6. $$\cos A\cos B\cos C\sum\frac{a}{\cos A}=2R\cos A\cos B\cos C\sum\tan A=2R\cos A\cdot\cos B\cdot\cos C\cdot\prod(\tan A)$$

$$= 2R\cos A\cdot\cos B\cdot\cos C\cdot\frac{\sin A\sin B\sin C}{\cos A\cos B\cos C} = 2R\sin A\sin B\sin C$$

8. In ΔBAD, $\dfrac{BD}{\sin\theta} = \dfrac{AD}{\sin 60^\circ}$

In ΔCAD, $\dfrac{CD}{\sin(75^\circ-\theta)} = \dfrac{AD}{\sin 45^\circ}$



$$\Rightarrow \quad \frac{BD}{\sin\theta}\sin 60^\circ = \frac{CD\sin 45^\circ}{\sin(75^\circ-\theta)}$$

$$\Rightarrow \quad \frac{\sin\theta}{\sin(75^\circ-\theta)} = \frac{BD}{CD}\frac{\sin 60^\circ}{\sin 45^\circ} = \frac{1}{\sqrt{6}}$$

9. Length of angle bisector $AD = \dfrac{2bc}{b+c}\cos\dfrac{A}{2}$

Length of angle bisector $BE = \dfrac{2ac}{a+c}\cos\dfrac{B}{2}$

Length of angle bisector $CF = \dfrac{2ab}{a+b}\cos\dfrac{C}{2}$

$$\text{H.M.} = \frac{3}{\dfrac{b+c}{2bc}+\dfrac{a+c}{2ac}+\dfrac{a+b}{2ab}} = \frac{3}{\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}}$$

10. $2b = a + c$

$2\sin B = \sin A + \sin C$

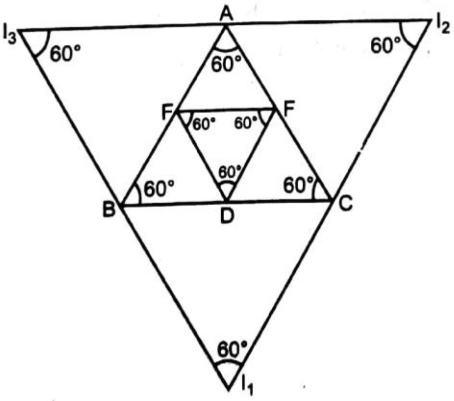
$$2\left(2\sin\frac{B}{2}\cos\frac{B}{2}\right) = 2\sin\left(\frac{A+C}{2}\right)\cos\left(\frac{A-C}{2}\right) \Rightarrow \sin\frac{B}{2} = \frac{1}{2\sqrt{2}}$$

11. $2\cos\left(\dfrac{B-C}{2}\right) = \dfrac{b+c}{a} = \dfrac{\sin B+\sin C}{\sin A}$

$\Rightarrow \quad \sin\dfrac{A}{2} = \dfrac{1}{2} \quad \Rightarrow \quad \angle A = 60^\circ$

12. $\cos A = \dfrac{4+c^2-1}{4c} = \dfrac{1}{4}\left(c+\dfrac{3}{c}\right) \geq \dfrac{\sqrt{3}}{2}$

13.



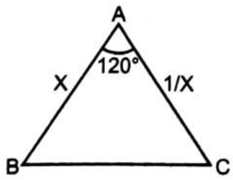
If ΔABC is an equilateral triangle then ΔDEF and $\Delta\, I_1 I_2 I_3$ are also equilateral triangle

Side of $\Delta DEF = 1$ unit $\Rightarrow$ $Ar(\Delta DEF) = \frac{\sqrt{3}}{4}$

14.

$$AD = \frac{2x\cdot\frac{1}{x}}{x+\frac{1}{x}}\cos\frac{\pi}{3} = \frac{1}{x+\frac{1}{x}}$$

$$AD_{\max} = \frac{1}{2}$$



15. $r = \frac{\sqrt{3}a}{6}, R = \frac{\sqrt{3}a}{3}, r_1 = \frac{\sqrt{3}a}{2} \Rightarrow r, R, r_1$ are in A.P.

16. $\sin(B+C)\sin(B-C) = \sin(A+B)\sin(A-B)$

$$\sin^2 B - \sin^2 C = \sin^2 A - \sin^2 B$$

$$\Rightarrow \quad 2\sin^2 B = \sin^2 A + \sin^2 C$$

$$\Rightarrow \quad 2b^2 = a^2 + c^2 \qquad \text{(Using sine rule)}$$

17.

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \tan(\pi - C)$$

$$\Rightarrow \quad \tan C = \frac{7}{4} \Rightarrow \sin C = \frac{7}{\sqrt{65}}$$

Using sine rule

$$R = \frac{c}{2\sin C} = \frac{65}{14}$$

18. $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$

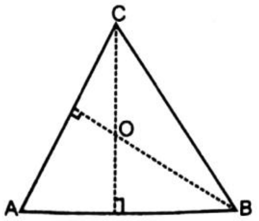
$$\frac{b^2+c^2-a^2}{2abc} + \frac{a^2+c^2-b^2}{2abc} + \frac{a^2+b^2-c^2}{2abc} = \frac{a^2+b^2+c^2}{2abc} = \frac{(a+b+c)^2 - 2(ab+bc+ac)}{2abc}$$

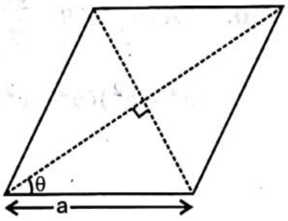
19. $\frac{a+c}{b} + \frac{b+c}{a} = \frac{a^2+b^2+ac+bc}{ab} = \frac{c^2(a+b+c)}{abc} = \frac{c^2(2s)}{4R\Delta} = \frac{2R}{r} = \frac{c}{r}$

20. $a^2(\sin B - 1) = b^2 + c^2 - a^2 = 2bc\cos A \Rightarrow \cos A < 0$

21. $2R' = \frac{a}{\sin(\pi - A)} = \frac{a}{\sin A} = 2R$

$$\Rightarrow \quad R' = R$$



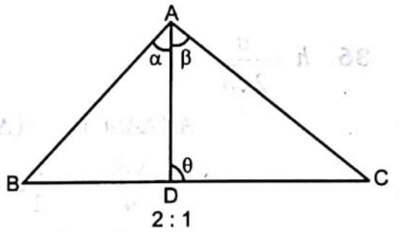


22. $$a=\sqrt{d_1 d_2}$$

$$\frac{d_1}{2}=a\cos\theta,\ \frac{d_2}{2}=a\sin\theta$$

$$1=4\sin\theta\cos\theta$$

$$\Rightarrow\ \sin 2\theta=\frac{1}{2}\Rightarrow 2\theta=30^\circ$$



23. $\because\ (m+n)\cot\theta=m\cot\alpha-n\cot\beta$

$\therefore\ (2+1)\cot\theta=2\cot\alpha-\cot\beta$

Put $\cot\theta=\dfrac{1}{3},\cot\beta=\cot\left(\dfrac{\pi}{2}-\alpha\right)=\tan\alpha$

We have $1=\dfrac{2}{\tan\alpha}-\tan\alpha\ \Rightarrow\ \tan^2\alpha+\tan\alpha-2=0$

$\therefore\ \tan\alpha=1\qquad\therefore\ \alpha=45^\circ$

24. Circumradius of equilateral $\Delta, R=\dfrac{l}{2\sin 60^\circ}=\dfrac{l}{\sqrt{3}}$

Diagonal of square $=2R\Rightarrow a\sqrt{2}=2R\quad\therefore\ a=R\sqrt{2}=\dfrac{l\sqrt{2}}{\sqrt{3}}\quad\therefore$ Area of square $=\dfrac{2l^2}{3}$

25. $$\cos\theta=\frac{2^2+(\sqrt{6})^2-(\sqrt{3}+1)^2}{4\sqrt{6}}=\frac{\sqrt{3}-1}{2\sqrt{2}}$$

26. If a,b,c are in A.P.

$$\Rightarrow\ 2\sin B=\sin A+\sin C\Rightarrow\sin\frac{B}{2}=\frac{1}{4}$$

$$\frac{s}{r}=\frac{6\cos\frac{B}{2}}{1-2\sin\frac{B}{2}}=3\sqrt{15}$$

27. $$\cos(A-B)=\frac{1-\tan^2\left(\frac{A-B}{2}\right)}{1+\tan^2\left(\frac{A-B}{2}\right)}=\frac{31}{32}\Rightarrow\tan\left(\frac{A-B}{2}\right)=\frac{1}{3\sqrt{7}}$$

$$\tan\left(\frac{A-B}{2}\right)=\frac{a-b}{a+b}\cot\frac{C}{2}\quad\Rightarrow\cos C=\frac{1}{8}$$

$$\cos C=\frac{a^2+b^2-c^2}{2ab}=\frac{1}{8}\quad\Rightarrow c=6$$

28. $(b+c)\cos(B+C)+(c+a)\cos(C+A)+(a+b)\cos(A+B)$

$=-[(b\cos A+a\cos B)+(c\cos A+a\cos C)+(c\cos B+b\cos C)]=-[a+b+c]=-30$

30. $\angle A=\frac{\pi}{7}, \angle B=\frac{2\pi}{7}, \angle C=\frac{4\pi}{7}$

$$(a^2-b^2)(b^2-c^2)(c^2-a^2)=a^2b^2c^2\left(1-\frac{b^2}{a^2}\right)\left(1-\frac{c^2}{b^2}\right)\left(1-\frac{a^2}{c^2}\right)$$

$$=a^2b^2c^2\left(1-\frac{\sin^2\frac{2\pi}{7}}{\sin^2\frac{\pi}{7}}\right)\left(1-\frac{\sin^2\frac{4\pi}{7}}{\sin^2\frac{2\pi}{7}}\right)\left(1-\frac{\sin^2\frac{\pi}{7}}{\sin^2\frac{4\pi}{7}}\right)=a^2b^2c^2$$

36. $h=\frac{a}{2\sqrt{3}}$

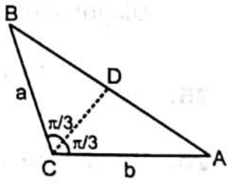
$$Ar(\Delta ABC)=Ar(\Delta APB)+Ar(\Delta BPC)+Ar(\Delta APC)$$

$$\frac{\sqrt{3}}{4}a^2=\frac{1}{2}a(h+h_1+h_2) \quad \Rightarrow \quad h_1+h_2=\frac{a}{\sqrt{3}}$$

37. $\cos 60° = \frac{6^2+7^2-x^2}{2\times 6\times 7} \Rightarrow x=\sqrt{43}$

39. $CD=\frac{2ab}{a+b}\cos\frac{\pi}{3}$

$=\frac{ab}{a+b}$



42. $a+b+c=48$

$a=20$

$b+c=28$

$\Rightarrow \quad a+b>c, a+c>b$

$\Rightarrow \quad 20+b>28-b,\ 20+c>28-c$

$\Rightarrow \quad b>4, c>4$

43. In an equilateral triangle

$a=b=c$

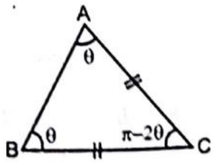
44. $\frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2}$

$$=\frac{2s(2s-2a)(2s-2b)(2s-2c)}{4b^2c^2}=4\left(\frac{2(s-a)}{bc}\right)\left(\frac{(s-b)(s-c)}{bc}\right)=4\sin^2\frac{A}{2}\cos^2\frac{A}{2}=\sin^2 A$$

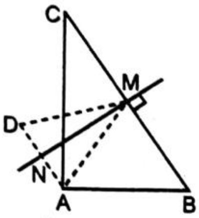
45. $R=4r$

$R=4\left(4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$

$1=16\sin^2\frac{\theta}{2}\cdot\cos\theta=8(1-\cos\theta)\cos\theta$



46. $\Delta DMN \cong \Delta AMN \Rightarrow DM = AM$



47. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^2 + c^3 - 3abc)$

$$= -(a+b+c)(a^2+b^2+c^2-ab-bc-ac)$$

48. $A = \frac{\pi}{7}, B = \frac{2\pi}{7}, C = \frac{4\pi}{7}$

$$\left(1-\frac{b^2}{a^2}\right)\left(1-\frac{c^2}{b^2}\right)\left(1-\frac{a^2}{c^2}\right) = \lambda$$

$$\left(1-\frac{\sin^2\frac{2\pi}{7}}{\sin^2\frac{\pi}{7}}\right)\left(1-\frac{\sin^2\frac{4\pi}{7}}{\sin^2\frac{2\pi}{7}}\right)\left(1-\frac{\sin^2\frac{\pi}{7}}{\sin^2\frac{4\pi}{7}}\right) = \lambda$$

$$\left(\frac{\sin^2\frac{\pi}{7}-\sin^2\frac{2\pi}{7}}{\sin^2\frac{\pi}{7}}\right)\left(\frac{\sin^2\frac{2\pi}{7}-\sin^2\frac{4\pi}{7}}{\sin^2\frac{2\pi}{7}}\right)\left(\frac{\sin^2\frac{4\pi}{7}-\sin^2\frac{\pi}{7}}{\sin^2\frac{4\pi}{7}}\right) = \lambda$$

$\Rightarrow \lambda = 1(\sin^2 A - \sin^2 B = \sin(A-B)\cdot\sin(A+B))$

49. $r_1 = \frac{\Delta}{s-a}, \quad r_2 = \frac{\Delta}{s-b}, \quad r_3 = \frac{\Delta}{s-c}$

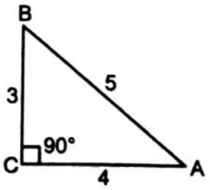
$$\frac{r_1 r_2 r_3}{r^3} = \frac{s^3}{(s-a)(s-b)(s-c)}$$

$$\frac{\frac{s-a}{s}+\frac{s-b}{s}+\frac{s-c}{s}}{3} \geq \left(\frac{s-a}{s}\right)\left(\frac{s-b}{s}\right)\left(\frac{s-c}{s}\right)$$

50. $\sin A = \frac{3}{5}$

$\sin B = \frac{4}{5}$

$\sin C = 1$



51. $$\frac{r_1+r_2}{1+\cos C}=\frac{4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}+4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}}{2\cos^2\frac{C}{2}}$$

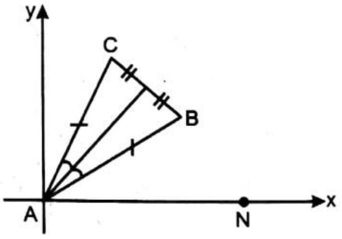
$$=\frac{2R\left(\sin\frac{A}{2}\cos\frac{B}{2}+\cos\frac{A}{2}\sin\frac{B}{2}\right)}{\cos\frac{C}{2}}=2R$$

53. $$\cos\theta=\frac{\sin^2\alpha+\cos^2\alpha-(1+\sin\alpha\cos\alpha)}{2\sin\alpha\cos\alpha}=\frac{1}{2}$$

55. Since we need to compute the radius of an escribed circle, we would be needing the length of all the sides of the given triangle ABC.

From the question, we already know $AB = AC = 5$.

For finding the length of side BC, let us draw a line AD which is the bisector of angle BAC, as shown in the figure below.



$$\angle BAD=\angle DAC=15^\circ$$

Therefore, $\sin 15^\circ=\frac{BD}{AB}=\frac{BD}{5}$ and $\sin 15^\circ=\frac{\sqrt{3}-1}{2\sqrt{2}}$

Therefore, $BD=5\sin 15^\circ=\frac{5(\sqrt{3}-1)}{2\sqrt{2}}$

We also know that $BC=2BD$

Therefore, $BC=\frac{5(\sqrt{3}-1)}{\sqrt{2}}$

Now, we know that the required radius

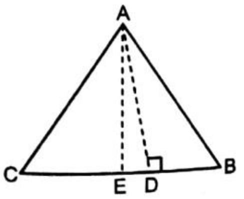
$$r_1=s\tan\left(\frac{A}{2}\right)=\left(\frac{AB+BC+CA}{2}\right)\tan\left(\frac{A}{2}\right)$$

$$=\left(\frac{5+\frac{5(\sqrt{3}-1)}{\sqrt{2}}+5}{2}\right)(\tan 15^\circ)=\left(\frac{10\sqrt{2}+5\sqrt{3}-5}{2\sqrt{2}}\right)(2-\sqrt{3})$$

56. $$ED=BE-BD=\frac{a}{2}-C\cos B$$

$$=\frac{a}{2}-C\left(\frac{a^2+c^2-b^2}{2ac}\right)$$

$$=\frac{b^2-c^2}{2a}$$

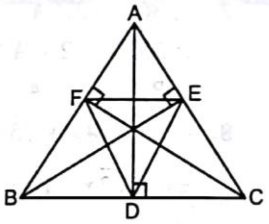


57. $2R(\sin A\cos B\cos C+\cos A\sin B\cos C+\cos A\cos B\sin C)$

$$=2R(\sin(A+B)\cos C+\cos A\cos B\sin C)$$

$$=R(2\sin A\sin B\sin C)=\frac{abc}{4R^2}=\frac{\Delta}{R}=\frac{rs}{R}$$

58. In ΔAFE, $\dfrac{b\cos A}{\sin B}=2R_1$



$\Rightarrow$ $R_1=R\cos A$

Similarly, $R_2=R\cos B$

and $R_3=R\cos C$

$$R_1+R_2+R_3=R(\cos A+\cos B+\cos C)\le\frac{3}{2}R$$

59. Ar (ΔABC) = Ar (ΔOAB) + Ar (ΔOBC) + Ar (ΔOAC)

$$8=\frac{1}{2}R^2(\sin\alpha+\sin\beta+\sin\gamma)$$

$$\Rightarrow\quad \sin\alpha+\sin\beta+\sin\gamma=\frac{4\pi}{5}\qquad\left(\because R^2=\frac{20}{\pi}\right)$$

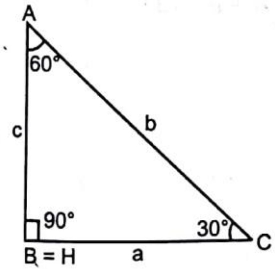
Exercise-2 : One or More than One Answer is/are Correct

1. $x^2-r(r_1r_2+r_2r_3+r_1r_3)x+(r_1r_2r_3-1)=0$

$$x^2-(r_1r_2r_3)x+(r_1r_2r_3-1)=0$$

$\Rightarrow$ Roots are 1 and $r_1r_2r_3-1$

2.



3. $R=2r, r=(s-a)\tan 30°=\dfrac{s}{3}\tan 30°\Rightarrow s$ is irrational $\Rightarrow\Delta$ is irrational

$$r_1=s\tan 30°=3r\ \text{(rational)}$$

4. $D+E+F=\dfrac{\pi}{2}$

5. $a=4, b=8, \angle C=60°$

$$\cos C=\frac{1}{2}=\frac{a^2+b^2-c^2}{2ab}\Rightarrow c=4\sqrt{3}$$

6. If $\dfrac{r}{r_1} = \dfrac{r_2}{r_3} \Rightarrow \dfrac{s-a}{s} = \dfrac{s-c}{s-b}$

$\Rightarrow \quad a^2 + b^2 = c^2$

$\Rightarrow \quad \angle C = 90°$

7. $\angle BOC = 2\angle A$

$\angle BIC = \pi/2 + A/2$

$\angle BHC = \pi - A$

8. $\sqrt{3}x^2 - 4x + \sqrt{3} < 0$

$\Rightarrow \quad (\sqrt{3}x - 1)(x - \sqrt{3}) < 0 \qquad \Rightarrow \dfrac{1}{\sqrt{3}} < x < \sqrt{3}$

$30° < A, B < 60° \qquad \Rightarrow 60° < C < 120°$

9. $\cos 2\theta = 2\cos^2\theta - 1$

$$\frac{1}{\sqrt{2}} = 2\cos^2\frac{\pi}{8} - 1$$

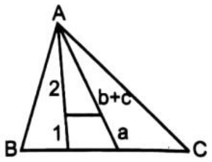
$$2\cos^2\frac{\pi}{8} = 1 + \frac{1}{\sqrt{2}}$$

$$\cos^2\frac{\pi}{8} = \frac{\sqrt{2}+1}{2\sqrt{2}}$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab} \text{ then solve it}$$

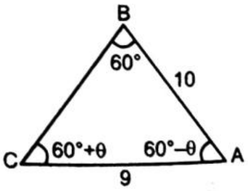
11. $(3\sin A + 4\cos B)^2 + (4\sin B + 3\cos A)^2 = 37; \quad 9 + 16 + 24\sin(A+B) = 37$

12. $\dfrac{b+c}{a} = \dfrac{2}{1}$



13. $\angle A, \angle B, \angle C$ A.P. $\Rightarrow \angle B = 60°$

$\cos 60° = \dfrac{a^2 + 10^2 - 9^2}{20a}$



14. $\Delta = \dfrac{1}{2}ab\sin C$

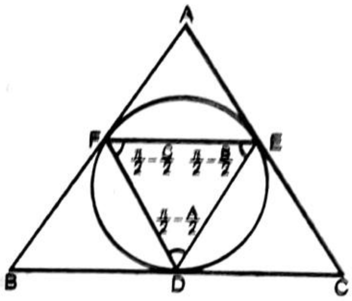
$\dfrac{a+b}{2} \geq \sqrt{ab} \quad \Rightarrow \quad \dfrac{\sin A + \sin B}{2} \geq \sqrt{\sin A \times \sin B}$

15. $3\cos A = \cos(B - C) - \cos(B + C) \Rightarrow 2\cos A = \cos B - C) = -\cos(A + 2C)$

$2 = (\tan A \sin 2C - \cos 2C)$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 2


A
B
C
D
E
F
$\frac{\pi}{2}-\frac{C}{2}$
$\frac{\pi}{2}-\frac{B}{2}$
$\frac{\pi}{2}-\frac{A}{2}$

2. $r' = 4r\sin\left(\frac{\pi}{4} - \frac{A}{4}\right)\sin\left(\frac{\pi}{4} - \frac{B}{4}\right)\sin\left(\frac{\pi}{4} - \frac{C}{4}\right)$

$$\frac{r'}{r} = 4\sin\left(\frac{\pi}{4} - \frac{A}{4}\right)\sin\left(\frac{\pi}{4} - \frac{B}{4}\right)\sin\left(\frac{\pi}{4} - \frac{C}{4}\right)$$

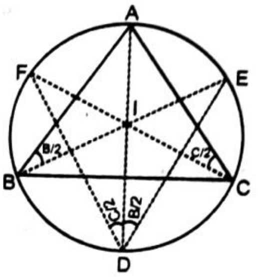
$$= \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} - 1$$

$$r_1' = 4r\sin\left(\frac{\pi}{4} - \frac{A}{4}\right)\cos\left(\frac{\pi}{4} - \frac{B}{4}\right)\cos\left(\frac{\pi}{4} - \frac{C}{4}\right)$$

$$\frac{r_1'}{r} = 4\sin\left(\frac{\pi}{4} - \frac{A}{4}\right)\cos\left(\frac{\pi}{4} - \frac{B}{4}\right)\cos\left(\frac{\pi}{4} - \frac{C}{4}\right)$$

$$= 1 - \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}$$

Paragraph for Question Nos. 3 to 4



3. $Ar(\Delta DEF)$

$$= 2R^2\sin\left(\frac{\pi}{2} - \frac{A}{2}\right)\sin\left(\frac{\pi}{2} - \frac{B}{2}\right)\sin\left(\frac{\pi}{2} - \frac{C}{2}\right)$$

$$= 2R^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

4. $\frac{Ar(\Delta ABC)}{Ar(\Delta DEF)} = \frac{2R^2\sin A\sin B\sin C}{2R^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le 1$

Paragraph for Question Nos. 5 to 6

Sol. $c/2 = R \Rightarrow c = 82$

$$2r = a + b - c$$

$$\Rightarrow \quad a + b = 98 \qquad \ldots(1)$$

$$a^2 + b^2 = c^2 = (82)^2 \qquad \ldots(2)$$

$$\Rightarrow \quad a = 18, b = 80$$

Paragraph for Question Nos. 7 to 8

Sol. $\angle A_1 = \frac{\pi}{2} - \frac{A}{2}$

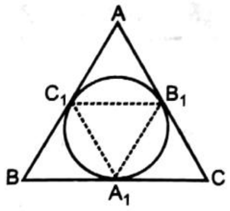
$$\angle A_2 = \frac{\pi}{2} - \frac{1}{2}(\angle A_1) = \frac{\pi}{2} - \frac{1}{2}\left(\frac{\pi}{2} - \frac{A}{2}\right)$$

$$= \frac{\pi}{4} + \frac{A}{4}$$

$$\angle A_3 = \frac{\pi}{2} - \frac{1}{2}(\angle A_2)$$

$$= \frac{3\pi}{8} - \frac{A}{8}$$

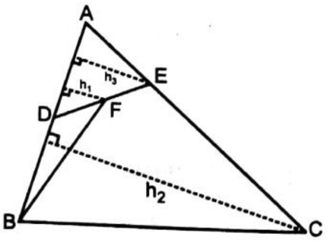
$$\angle A_n = \frac{\pi}{2}\left(1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots\ldots\right) + \frac{(-1)^n A}{2^n}$$



Paragraph for Question Nos. 9 to 10

Sol. $\frac{\Delta_1}{\Delta} = \frac{\frac{1}{2} \times BD \times h_1}{\frac{1}{2} \times AB \times h_2} = (1-x)\frac{h_1}{h_2} \times \frac{h_3}{h_3} = (1-x)\,yz$

$$\frac{\Delta_2}{\Delta} = \frac{\frac{1}{2} \times EC \times h_4}{\frac{1}{2} \times AC \times h_5} = x(1-y)(1-z)$$



Paragraph for Question Nos. 11 to 13

Sol. $\log\left(1 + \frac{c}{a}\right) + \log a - \log b = \log 2$

$\Rightarrow$ $a + c = 2b$

$(c-a)x^2 + 2bx + (a+c) = 0$ has equal roots, then

$$a^2 + b^2 = c^2$$

Paragraph for Question Nos. 14 to 16

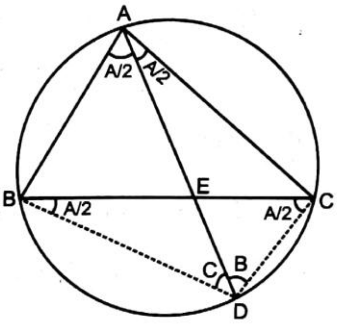
Sol. $\frac{BE}{\sin C} = \frac{ED}{\sin\frac{A}{2}}, \frac{EC}{\sin B} = \frac{ED}{\sin\frac{A}{2}}$

$$\Rightarrow \quad BE + EC = a = \frac{ED}{\sin\frac{A}{2}}(\sin B + \sin C)$$

$$\Rightarrow \quad ED = \frac{a\sin\frac{A}{2}\times 2R}{b+c}$$

$$\frac{l_a = \frac{2bc}{b+c}\cos\frac{A}{2}}{\left(\frac{2bc}{b+c}\cos\frac{A}{2} + \frac{a\sin\frac{A}{2}\times 2R}{b+c}\right)}$$

$$\Rightarrow \quad l_a = \frac{2\sin B\sin C}{2\sin B\sin C + 2\sin^2\frac{A}{2}} = \frac{\sin B\sin C}{\sin^2\left(B+\frac{A}{2}\right)}$$



Exercise-4 : Matching Type Problems

2. (A) $3^0\{2^0 + 2^{-1} + 2^{-2} \ldots\ldots\ldots\infty\} = 1\{2\}$

$3^{-1}\{2^0 + 2^{-1} + 2^{-2} \ldots\ldots\ldots\infty\} = \frac{1}{3}\{2\}$

$3^{-2}\{2^0 + 2^{-1} + 2^{-2} \ldots\ldots\ldots\infty\} = \frac{1}{3}\{2\}$

$\vdots \qquad\qquad \vdots$

∞

Hence, $\quad \frac{2\times 1}{1-\frac{1}{3}} = 3$

(B) $b^2 + c^2 - a^2 = 2bc\cos A = 54$

$bc\cos A = 27 = a^3 \Rightarrow a = 3$

$$\frac{b^2+c^2}{9} = \frac{63}{9} = 7$$

(C) Circumcentre of ΔABC is $(-1, 0)$.

Point A lie on the circle $(x+1)^2 + y^2 = 4 \Rightarrow x^2 + y^2 + 2x - 3 = 0$

(D) $(\cos\theta\sin\theta + 6) = 6(\sin\theta - \cos\theta) \Rightarrow 36 + \sin^2\theta\cos^2\theta + 12\sin\theta\cos\theta = 36(1 - 2\sin\theta\cos\theta)$

Let $\sin\theta\cos\theta = t$

$$t^2 + 84t = 0 \Rightarrow t = 0$$

If $\sin\theta = 0 \Rightarrow \cos\theta = -1 \Rightarrow \theta = \pi$

If $\cos\theta = 0 \Rightarrow \sin\theta = 1 \quad \Rightarrow \theta = \dfrac{\pi}{2}$

3. $r_1r_2 + r_3r_2 + r_1r_3 = S^2 \quad \Rightarrow \; S = 42$

$\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} = \dfrac{1}{r} \quad \Rightarrow \; r = 8$

$$r = \frac{\Delta}{S} \Rightarrow \Delta = 336$$

4. Use $r = \dfrac{\Delta}{s}$, $r_1 = \dfrac{\Delta}{s-a}$, $r_2 = \dfrac{\Delta}{s-b}$, $r_3 = \dfrac{\Delta}{s-a}$

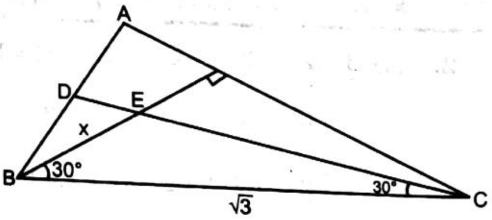
(C) $r = 4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

and similarly r_1, r_2, r_3

Exercise-5 : Subjective Type Problems

2. $\angle O_1EO_2 = 90°$, E is the orthocentre of ΔO_1EO_2

$\dfrac{x}{\sin 30°} = \dfrac{\sqrt{3}}{\sin 120°}; x = 1$

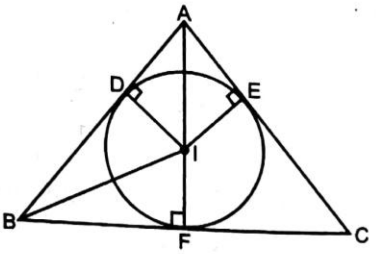


3. $\dfrac{1}{2}r(AD + AE) = 5$

$\dfrac{1}{2}r(BF + BD) = 10$

$\Rightarrow \quad \dfrac{BF + BD}{AD + AE} = 2 \Rightarrow \dfrac{r\cot\dfrac{B}{2} + r\cot\dfrac{B}{2}}{r\cot\dfrac{A}{2} + r\cot\dfrac{A}{2}} = 2$

Applying C and D, $\dfrac{\cos\dfrac{C}{2}}{\sin\dfrac{A-B}{2}} = 3$



4. $\dfrac{\Delta_1\Delta_2\Delta_3}{\Delta^3} = \dfrac{(r_1r_2r_3)^2}{r^6} = \left(\dfrac{s}{s-a} \times \dfrac{s}{s-b} \times \dfrac{s}{s-c}\right)^2$

$\dfrac{(s-a)+(s-b)+(s-c)}{3} \geq [(s-a)(s-b)(s-c)]^{1/3}$

$$\Rightarrow \quad \frac{s^3}{(s-a)(s-b)(s-c)} \geq 27$$

Minimum value = 1

5. In ΔABM, $\quad \dfrac{AB}{\sin 150^\circ} = \dfrac{AM}{\sin 7^\circ}$

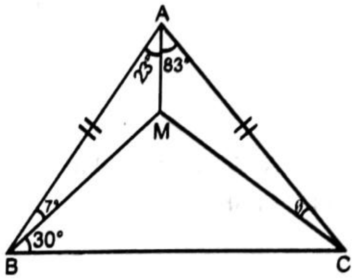
In ΔACM, $\quad \dfrac{AC}{\sin(97^\circ - \theta)} = \dfrac{AM}{\sin\theta}$

$$\Rightarrow \quad \sin\theta = 2\sin 7^\circ \sin(97^\circ - \theta)$$

$$\Rightarrow \quad \sin\theta = \sin\theta - \cos(104^\circ - \theta)$$

$$\Rightarrow \quad \cos(104^\circ - \theta) = 0$$

$$\Rightarrow \quad \theta = 14^\circ$$



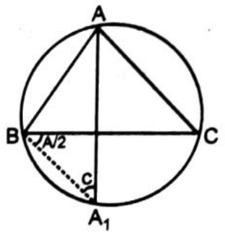
6. $\dfrac{AH}{AD} + \dfrac{BH}{BE} + \dfrac{CH}{CF} = \dfrac{R}{\Delta}(a\cos A + b\cos B + c\cos C) = \dfrac{R^2}{\Delta}(\sin 2A + \sin 2B + \sin 2C)$

$$= \frac{4R^2}{\Delta}\sin A \sin B \sin C = \frac{bc\sin A}{\Delta} = 2$$

7. $\dfrac{c}{\sin C} = \dfrac{AA_1}{\sin\left(B + \dfrac{A}{2}\right)}$

$AA_1 \cos\dfrac{A}{2} = \sin B + \sin C \qquad (\because R = 1)$

$$\Rightarrow \quad \frac{AA_1 \cos\frac{A}{2} + BB_1 \cos\frac{B}{2} + CC_1 \cos\frac{C}{2}}{\sin A + \sin B + \sin C} = 2$$



8. $ax^2 + bx + c = 0$ has equal roots, then

$$b^2 = 4ac \qquad \ldots(1)$$

$$\frac{\sin A}{\sin C} + \frac{\sin C}{\sin A} = \frac{a}{c} + \frac{c}{a} = \frac{a^2 + c^2}{ac} = \frac{b^2 + 2ac\cos B}{ac}$$

$$= 4 + 2\cos B$$

9. $\cot\dfrac{A}{2}, \cot\dfrac{B}{2}, \cot\dfrac{C}{2}$ is AP

In ΔABC,

$$\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2} \cdot \cot\frac{B}{2} \cdot \cot\frac{C}{2}$$

$$\Rightarrow \quad \cot\frac{A}{2} \cdot \cot\frac{C}{2} = 3$$

AM ≥ GM

$$\frac{\cot\frac{A}{2}+\cot\frac{C}{2}}{2}\ge\sqrt{\cot\frac{A}{2}\cdot\cot\frac{C}{2}}\quad\Rightarrow\ \cot\frac{B}{2}\ge\sqrt{3}$$

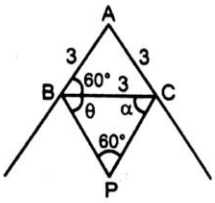
10. $(R^2-4Rr+4r^2)+(4r^2-12r+9)=0$

$$(R-2r)^2+(2r-3)^2=0$$

$\Rightarrow$ $$r=\frac{3}{2}\ ;\ R=2r$$

ΔABC is an equilateral triangle.

11. In ΔBCP,



$$\frac{3}{\sin 60^\circ}=\frac{PC}{\sin\theta}$$

$$PC=2\sqrt{3}\sin\theta$$

12. $b+c=\dfrac{2ab\cos C+2\sqrt{3}ab\sin C}{2b}=\dfrac{(a^2+b^2-c^2)+12}{2b}$

13. $R=3,\Delta=6$

$$P_{\Delta DEF}=DE+EF+DF=R(\sin 2A+\sin 2B+\sin 2C)$$

$$=4R\sin A\sin B\sin C$$

$$=4R\left(\frac{b}{2R}\frac{c}{2R}\sin A\right)=\frac{1}{R}(2\Delta)=4$$

* * *

Chapter 25 – Inverse Trigonometric Functions

25

INVERSE TRIGONOMETRIC FUNCTIONS

Exercise-1 : Single Choice Problems

2. $(\cot^{-1} x)\left(\frac{\pi}{2} - \cot^{-1} x\right) + 2\cot^{-1} x - \frac{\pi}{2}\cot^{-1} x + 3\left(\frac{\pi}{2} - \tan^{-1} x\right) - 6 > 0$

$$-(\cot^{-1} x)^2 + 5\cot^{-1} x - 6 > 0$$

$$(\cot^{-1} x)^2 - 5(\cot^{-1} x) + 6 < 0$$

$$(\cot^{-1} x - 3)(\cot^{-1} x - 2) < 0$$

$$2 < \cot^{-1} x < 3$$

$$\cot 3 < x < \cot 2$$

($\because$ $\cot^{-1} x$ is decreasing)

3. $1 + \tan^2(\tan^{-1} 2) + 1 + \cot^2(\cot^{-1} 3) = 1 + 2^2 + 1 + 3^2 = 15$

4. $\sum_{n=1}^{\alpha} \tan^{-1}\left(\frac{(n+1)^2 + (n+1) - ((n+1)^2 - (n+1))}{1 + (n+1)^4 - (n+1)^2}\right)$

5. $\cot^{-1}(\sqrt{\cos\alpha}) - \tan^{-1}(\sqrt{\cos\alpha}) = x$

$$\frac{\pi}{2} - 2\tan^{-1}\sqrt{\cos\alpha} = x$$

$$\frac{\pi}{4} - \frac{x}{2} = \tan^{-1}\sqrt{\cos\alpha}$$

$$\sqrt{\cos\alpha} = \frac{1 - \tan\frac{x}{2}}{1 + \tan\frac{x}{2}}$$

$$\Rightarrow \qquad \tan\frac{x}{2} = \frac{1 - \sqrt{\cos\alpha}}{1 + \sqrt{\cos\alpha}} \quad \Rightarrow \sin x = \tan^2\frac{\alpha}{2}$$

6. $T_n = \tan^{-1}\left(\frac{4}{4n^2+3}\right) = \tan^{-1}\left(\frac{1}{n^2+(3/4)}\right) = \tan^{-1}\left(\frac{\left(n+\frac{1}{2}\right)-\left(n-\frac{1}{2}\right)}{1+\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}\right)$

$T_n = \tan^{-1}\left(n+\frac{1}{2}\right) - \tan^{-1}\left(n-\frac{1}{2}\right)$

$S_n = \tan^{-1}\left(n+\frac{1}{2}\right) - \tan^{-1}\left(\frac{1}{2}\right) \Rightarrow S_\infty = \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}\right)$

7. $\cos^{-1}(1-x) + m\cos^{-1}x = \frac{n\pi}{2}$

Domain $x \in [0, 1]$

$\cos^{-1}(1-x) + m\cos^{-1}x > 0 \quad (\because m > 0)$

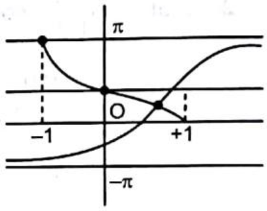
There is no solution.

8. $2\tan^{-1}(2x-1) = \cos^{-1}x$

$2x - 1 \geq 0 \qquad 1 \geq x > 0$

$x \geq \frac{1}{2}$

Only one solution



9. Put $x = 2\sin\theta,\ y = 3\cos\theta$

$$\frac{x}{2\sqrt{2}} + \frac{y}{3\sqrt{2}} - 2 = \frac{\sin\theta}{\sqrt{2}} + \frac{\cos\theta}{\sqrt{2}} - 2 \in [-3, -1]$$

$\therefore \quad \frac{\sin\theta}{\sqrt{2}} + \frac{\cos\theta}{\sqrt{2}} - 2 = -1$ only

10. $(\cos^{-1}x)^2 - (\sin^{-1}x)^2 > 0 \Rightarrow (\cos^{-1}x + \sin^{-1}x)(\cos^{-1}x - \sin^{-1}x) > 0$

$\Rightarrow \quad \cos^{-1}x - \sin^{-1}x > 0$

$\Rightarrow \quad \frac{\pi}{2} - 2\sin^{-1}x > 0 \Rightarrow -\frac{\pi}{2} \leq \sin^{-1}x < \frac{\pi}{4} \Rightarrow -1 \leq x < \frac{1}{\sqrt{2}}$

11. $f(x) = x^2 + 7x + k(k-3) = 0$

$f(0) < 0 \quad (\because k \in (0, 3))$

$\Rightarrow$ α and β are of opposite sign.

$\tan^{-1}\alpha + \tan^{-1}\left(\frac{1}{\alpha}\right) + \tan^{-1}\beta + \tan^{-1}\left(\frac{1}{\beta}\right) = 0$

12. $f(x) = a + 2b\cos^{-1}x$

$D_f : [-1, 1]$

$f(x)$ is decreasing function.

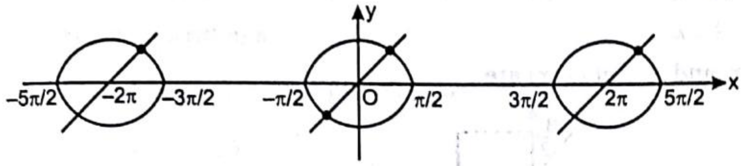
$\Rightarrow \quad f(-1) = 1 \quad \Rightarrow a + 2b\pi = 1$

and $f(1) = -1 \quad \Rightarrow a = -1$

13. Let $\tan^{-1} x = t \quad \Rightarrow t^2 + \left(\frac{\pi}{2} - t\right)^2 = \frac{5\pi^2}{8}$

$$\Rightarrow t = \frac{3\pi}{4} \text{ or } \frac{-\pi}{4} \quad \Rightarrow \tan^{-1} x = \frac{-\pi}{4} \quad \Rightarrow x = -1$$

14.



15. $1 \le \sin^{-1}(\cos^{-1}(\sin^{-1}(\tan^{-1} x))) \le \frac{\pi}{2}$

$\sin 1 \le \cos^{-1}(\sin^{-1}(\tan^{-1} x)) \le 1$

$\cos(\sin 1) \ge \sin^{-1}(\tan^{-1} x) \ge \cos 1$

$\sin(\cos(\sin 1)) \ge \tan^{-1} x \ge \sin(\cos 1)$

$\tan(\sin(\cos(\sin 1))) \ge x \ge \tan(\sin(\cos 1))$

16. $x + \frac{1}{x} = -2\sin(\cos^{-1} y) \quad \Rightarrow x = -1 \text{ and } y = 0$

17. $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3$

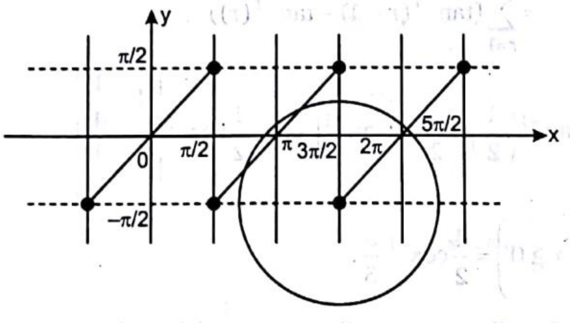
$\tan^{-1} 1 + \pi + \tan^{-1}\left(\frac{5}{1-6}\right) = \pi$

18. Let $\tan^{-1} x = \theta, \theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$2\theta + \cos^{-1}\cos 2\theta \Rightarrow 2\theta \le 0$

$\theta \le 0 \Rightarrow \tan^{-1} x \le 0 \Rightarrow x \le 0$

19.



$$16(x^2 + y^2) - 48\pi x + 16\pi y + 31\pi^2 = 0$$

$$x^2 + y^2 - 3\pi x + \pi y + \frac{31\pi^2}{16} = 0$$

$$\left(x-\frac{3\pi}{2}\right)^2+\left(y+\frac{\pi}{2}\right)^2=\frac{9\pi^2}{16}$$

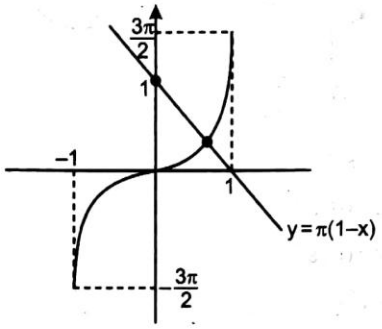
22. $\sin^{-1}(\sin 8)=3\pi-8=t$

$\tan^{-1}(\tan 8)=8-3\pi=-t$

$f(t)+f(-t)=\lambda$

$2=\lambda$

23. Graphs of $y=3\sin^{-1}x$ and $y=\pi(1-x)$ are



Clearly one point of intersection

24. $D_f:[-1,1]$

$f(x)_{\max}=\frac{\pi}{2}+6$ at $x=1$

$f(x)_{\min}=-\frac{\pi}{2}-2$ at $x=-1$

27. $\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right)+\ldots=\sum_{r=1}^{\infty}\tan^{-1}\left(\frac{1}{r^2+r+1}\right)$

$$=\sum_{r=1}^{\infty}(\tan^{-1}(r+1)-\tan^{-1}(r))$$

28. $\frac{1}{2}\cos^{-1}x=\tan^{-1}\frac{(1/4)+(2/9)}{1-(1/4)\times(2/9)}=\tan^{-1}\left(\frac{1}{2}\right)=\frac{1}{2}\times 2\tan^{-1}\left(\frac{1}{2}\right)=\frac{1}{2}\cos^{-1}\left[\frac{1-\frac{1}{4}}{1+\frac{1}{4}}\right]$

$\left(\text{using } 2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\text{ for } x\geq 0\right)=\frac{1}{2}\cos^{-1}\frac{3}{5}.$

29. $\tan^2(\sin^{-1}x)>1\Rightarrow-\frac{\pi}{2}<\sin^{-1}x<-\frac{\pi}{4}$ or $\frac{\pi}{4}<\sin^{-1}x<\frac{\pi}{2}$

30. $\cot^{-1}\left(\frac{1+2\times 4}{4-2}\right)+\cot^{-1}\left(\frac{1+4\times 8}{8-4}\right)+\cot^{-1}\left(\frac{1+8\times 16}{16-8}\right)+\ldots$

$$= \cot^{-1}(2) - \cot^{-1}(4) + \cot^{-1}(4) - \cot^{-1}(8) + \cot^{-1}(8) - \cot^{-1}(16) + \ldots$$

$$= \cot^{-1}(2)$$

32. $\sin^{-1}(1+x)$ is defined for $x < 0$ and $\sin^{-1} x = \frac{\pi}{2} - \cos^{-1} x \;\forall\; -1 \le x \le 1$.

The given equation is $\sin^{-1} x + \sin^{-1}(1+x) = \cos^{-1} x$

which can be written as

$$\frac{\pi}{2} - \cos^{-1} x + \frac{\pi}{2} - \cos^{-1}(1+x) = \cos^{-1} x$$

$$\Rightarrow \quad \pi - \cos^{-1}(1+x) = 2\cos^{-1} x$$

$$\Rightarrow \quad \cos^{-1}(-1-x) = 2\pi - \cos^{-1}(2x^2 - 1)$$

$$\Rightarrow \quad \cos^{-1}(-1-x) + \cos^{-1}(2x^2 - 1) = 2\pi$$

$$\Rightarrow \quad \cos^{-1}(-1-x) = \cos^{-1}(2x^2 - 1) = \pi$$

$$\Rightarrow \quad -1 - x = 2x^2 - 1 = -1$$

$$\Rightarrow \quad x = 0$$

which implies that the total number of solutions $\sin^{-1} x + \sin^{-1}(1+x) = \cos^{-1} x$ is only one.

33. $(\sin^{-1} x)^3 - (\cos^{-1} x)^3 + (\sin^{-1} x)(\cos^{-1} x)(\sin^{-1} x - \cos^{-1} x) = \frac{\pi^3}{16}$

$$(\sin^{-1} x - \cos^{-1} x)\{(\sin^{-1} x)^2 + (\cos^{-1} x)^2 + (2\cos^{-1} x \sin^{-1} x)\} = \frac{\pi^3}{16}$$

$$(\sin^{-1} x - \cos^{-1} x)(\sin^{-1} x + \cos^{-1} x)^2 = \frac{\pi^3}{16}$$

$$(\sin^{-1} x - \cos^{-1} x)\frac{\pi^2}{4} = \frac{\pi^3}{16}$$

$$2\sin^{-1} x - \frac{\pi}{2} = \frac{\pi}{4}$$

$$2\sin^{-1} x = \frac{3\pi}{4}$$

$$\sin^{-1} x = \frac{3\pi}{8}$$

$$x = \sin\frac{3x}{8} \text{ or } \cos\frac{x}{8}$$

35. $f(x) = \tan^{-1}\left(\frac{\sqrt{1+x^2} - 1}{x}\right)$

$$\frac{\sqrt{1+x^2} - 1}{x} = y$$

$$y' = \frac{x\frac{1}{2}\frac{2x}{\sqrt{1+x^2}} - (\sqrt{1+x^2}-1)}{x^2}$$

$$= \frac{\sqrt{1+x^2}-1}{x^2(\sqrt{1+x^2})} > 0 \quad \text{always}$$

$x \to \infty \qquad y \to 1$

$x \to -\infty \qquad y \to -1$

$\tan^{-1}(-1 \to 1)$

$\left(-\frac{\pi}{4}, \frac{\pi}{4}\right) - \{0\}$

40. $\cos^{-1} x + \cot^{-1} x = \lambda \; \forall \; x \in [-1, 1]$

$\lambda \in \left[\frac{\pi}{4}, \frac{7\pi}{4}\right]$

41. $x^3 + bx^2 + cx + 1 = 0$

$f(-1) = b - c < 0$

$f(0) = 1 > 0$

$\Rightarrow \; -1 < \alpha < 0$

$\alpha = -B$

$B \in (0, 1)$

$$y = -2\tan^{-1}(\operatorname{cosec} B) - \tan^{-1}\left(\frac{2\sin B}{\cos^2 B}\right)$$

$$= -\left(\pi + \tan^{-1}\frac{2\cos B}{1-\operatorname{cosec}^2 B}\right) - \tan^{-1}\frac{2\sin B}{\cos^2 B} = -\pi$$

42. $f(x) = \frac{\pi}{2} + \cot^{-1}\{-x\}$

$\frac{\pi}{4} < \cot^{-1}\{-x\} \le \frac{\pi}{2}$

43. $\sin^{-1}(\sin 3) + \tan^{-1}(\tan 3) + \sec^{-1}(\sec 3)$

$(\pi - 3) + (3 - \pi) + 3 = 3$

44. $(2n\pi, 0)\, n \in I$

45. $f(x) = \sin^{-1}([x] - 1) + 2\cos^{-1}([x] - 2)$

$-1 \le [x] - 1 \le 1 \quad \Rightarrow 0 \le [x] \le 2$

$-1 \le [x] - 2 \le 1 \quad \Rightarrow 1 \le [x] \le 3 \quad \Rightarrow [x] = 1 \text{ or } 2$

Exercise-2 : One or More than One Answer is/are Correct

2. $\cos^{-1} x = \tan^{-1} x \Rightarrow x \in [0, 1]$

$$\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \tan^{-1} x$$

$$\Rightarrow \qquad x^2 = \sqrt{1-x^2} \Rightarrow x^4 + x^2 - 1 = 0$$

$$x^2 = \frac{\sqrt{5}-1}{2}$$

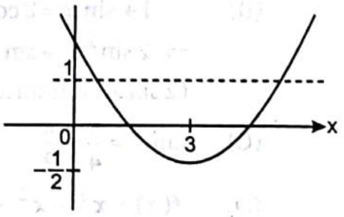
3. $\tan\left(\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{2}{3}\right) = \tan\left(\tan^{-1}\left(\frac{17}{6}\right)\right)$

$a = 17,\ b = 6$

5. $\sin^{-1}\left(x^2 - 6x + \frac{17}{2}\right) = \sin^{-1} k$

where $-1 \le k \le 1$

$$y = x^2 - 6x + \frac{17}{2}$$



6. $(\sin^{-1} x - \cos^{-1} x)((\sin^{-1} x)^2 + (\cos^{-1} x)^2 + 2\sin^{-1} x \cos^{-1} x) = \frac{\pi^3}{16}$

$$\Rightarrow \ \sin^{-1} x - \cos^{-1} x = \frac{\pi}{4} \ \Rightarrow \cos^{-1} x = \frac{\pi}{8} \ \Rightarrow x = \cos\frac{\pi}{8}$$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 2

1. $a = 2\pi$

$b = -3$

2. $a = 0$

$b = 3$

Exercise-4 : Matching Type Problems

3. (A) $33n = \frac{n}{2}[2 + (n-1)^2] \Rightarrow n = 9$

(B) $x \in [-1, 1] \Rightarrow \cos^{-1} x + \cot^{-1} x \in \left[\frac{\pi}{4}, \frac{7\pi}{4}\right]$

(C) $\cos\theta = |1 + \sin\theta| \Rightarrow \cos\theta \geq 0$

Sq. both sides,

$\Rightarrow \quad \cos^2\theta = 1 + \sin^2\theta + 2\sin\theta$

$\sin\theta = 0$ or $\sin\theta = -1$

Number of solution $= 3$

(D) $a = x(x-1)$

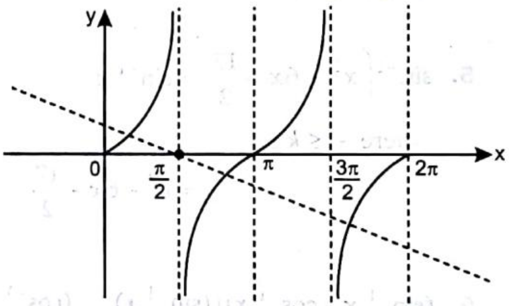
Possible values of a are 6, 12, 20, 30.

4. (A) $\tan^{-1}(3) + \tan^{-1}(-3) = 0$

(B) $1 + \sin x = 2\cos^2 x$

$\Rightarrow 2\sin^2 x + \sin x - 1 = 0$

$(2\sin x - 1)(\sin x + 1) = 0$



(C) $\tan x = \frac{\pi}{4} - \frac{\pi}{2}$

(D) $f(x) = x^3 + x^2 + 4x + 2\sin x$

$f'(x) = 3x^2 + 2x + 4 + 2\cos x > 0$

and $f(0) = 0$

Exercise-5 : Subjective Type Problems

1. $5 - 2\pi > x^2 - 4x$

$x^2 - 4x + (2\pi - 5) < 0$

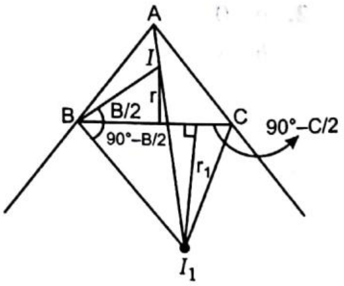
$2 - \sqrt{9 - 2\pi} < x < 2 + \sqrt{9 - 2\pi} \quad \Rightarrow \lambda = 9$

2.

$$\sin\frac{B}{2} = \frac{r}{IB}$$

$$IB = 4R\sin\frac{A}{2}\sin\frac{C}{2}$$

$$\sin\left(90° - \frac{B}{2}\right) = \frac{r_1}{BI_1} \Rightarrow BI_1 = 4R\sin\frac{A}{2}\cos\frac{C}{2}$$



$$(II_1)^2 = (BI)^2 + (BI_1)^2 = 16R^2\sin^2\frac{A}{2} \qquad \ldots(1)$$

$$I_2I_3\cos\left(90^\circ-\frac{A}{2}\right)=a \quad \text{(by using pedal triangle)}$$

$$I_2I_3=4R\cos\frac{A}{2}$$

$$(I_2I_3)^2=16R^2\cos^2\frac{A}{2} \qquad \ldots(2)$$

From (1) & (2) we get $\lambda=16$

3. $2\tan^{-1}\left(\frac{1}{5}\right)-\sin^{-1}\left(\frac{3}{5}\right)$

$$\tan^{-1}\left(\frac{5}{12}\right)-\sin^{-1}\left(\frac{3}{5}\right)$$

$$\tan^{-1}\left(\frac{5}{12}\right)-\tan^{-1}\left(\frac{3}{4}\right)=-\left(\tan^{-1}\left(\frac{3}{4}\right)-\tan^{-1}\left(\frac{5}{12}\right)\right)$$

$$=-\tan^{-1}\left(\frac{16}{63}\right)=-\cos^{-1}\left(\frac{63}{65}\right)$$

$\Rightarrow \qquad \lambda=65$

5. $$\sum_{n=0}^{\infty}2\tan^{-1}\left(\frac{2}{n^2+n+4}\right)=\sum_{n=0}^{\infty}2\tan^{-1}\left(\frac{\frac{1}{2}}{\frac{n^2}{4}+\frac{n}{4}+1}\right)$$

$$=\sum_{n=0}^{\infty}2\tan^{-1}\left(\frac{\left(\frac{n}{2}+\frac{1}{2}\right)-\frac{n}{2}}{\frac{n}{2}\left(\frac{n}{2}+\frac{1}{2}\right)+1}\right)$$

$$=\sum_{n=0}^{\infty}2\left(\tan^{-1}\left(\frac{n}{2}+\frac{1}{2}\right)-\tan^{-1}\left(\frac{n}{2}\right)\right)$$

6. $\cos^{-1}(|3\log_6^2(\cos x)-7|)=\cos^{-1}(|\log_6^2(\cos x)-1|)$

$$|3\log_6^2(\cos x)-7|=|\log_6^2(\cos x)-1|$$

Let $\log_6^2(\cos x)=t$

$$|3t-7|=|t-1|$$

$\Rightarrow \quad t=3$ and $t=2$

$\Rightarrow \quad \cos x=6^{-\sqrt{3}}$ and $6^{-\sqrt{2}}$

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Chapter 26 – Vector & 3Dimensional Geometry

26

VECTOR & 3DIMENSIONAL GEOMETRY

Exercise-1 : Single Choice Problems

1. Perpendicular distance from origin

$$d=\frac{p}{\sqrt{a^2+b^2+c^2}}$$

$$\therefore \quad d^2=\frac{p^2}{a^2+b^2+c^2}$$

2. Area of triangle $=\frac{1}{2}|\vec{a}\times\vec{b}|=3$

$$\Rightarrow \quad |\vec{a}||\vec{b}|\sin\frac{\pi}{3}=6 \quad \Rightarrow |\vec{a}||\vec{b}|=\frac{12}{\sqrt{3}}$$

$$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\frac{\pi}{3}=2\sqrt{3}$$

4. $|\vec{c}-\vec{a}|^2=8 \Rightarrow |\vec{c}|^2-2\,\vec{c}\cdot\vec{a}+|\vec{a}|^2=8 \Rightarrow |\vec{c}|^2-2|\vec{c}|+1=0 \Rightarrow |\vec{c}|^2=1$

Also, $\quad \vec{a}\times\vec{b}=2\hat{i}+2\hat{j}+\hat{k} \quad \Rightarrow |\vec{a}\times\vec{b}|=3$

$$\therefore \quad |(\vec{a}\times\vec{b})\times\vec{c}|=|\vec{a}\times\vec{b}||\vec{c}|\sin\frac{\pi}{6}=3\cdot1\cdot\frac{1}{2}=\frac{3}{2}$$

5.

$$\cos\theta_1=\frac{4}{5}$$

$$\cos\theta_2=\frac{4}{5}$$

$$\Rightarrow \quad \cos^2\theta_1+\sin^2\theta_2=1$$

7. $\lambda(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b})=4\sqrt{3} \qquad \left(ab\cos\frac{\pi}{3}=1\right) \Rightarrow \quad b=1$

$$\lambda(a^2b^2-(\vec{a}\cdot\vec{b})^2)=4\sqrt{3}$$

$$\lambda(4\times1-(1)^2)=4\sqrt{3}$$

$$\lambda=\frac{4\sqrt{3}}{3}$$

8. $x(3\hat{i}+2\hat{j}+4\hat{k})+y(2\hat{i}+2\hat{k})+z(4\hat{i}+2\hat{j}+3\hat{k})=\alpha(x\hat{i}+y\hat{j}+z\hat{k})$

$$\Rightarrow \quad (3-\alpha)x+2y+4z=0$$

$$2x-\alpha y+2z=0$$

$$4x+2y+(3-\alpha)z=0$$

For non-trivial solution

$$\begin{vmatrix}3-\alpha & 2 & 4\\ 2 & -\alpha & 2\\ 4 & 2 & 3-\alpha\end{vmatrix}=0$$

9. $\begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c}\\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c}\\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c}\end{vmatrix}=[\vec{a}\ \ \vec{b}\ \ \vec{c}]^2$

10. $|\vec{c}|^2=4(\vec{a}\times\vec{b})^2+9b^2=4(a^2b^2-(\vec{a}\cdot\vec{b})^2)+9b^2=192$

$\vec{c}+3\vec{b}=2\vec{a}\times\vec{b}\Rightarrow c^2+9b^2+6\vec{b}\cdot\vec{c}=4(a^2b^2-(\vec{a}\cdot\vec{b})^2)$

$\Rightarrow\quad 6\cdot4\cdot\sqrt{192}\cos\theta=-288\quad\Rightarrow\cos\theta=\frac{-\sqrt{3}}{2}$

11. $|\vec{a}-2\vec{b}|^2+|\vec{b}-2\vec{c}|^2+|\vec{c}-2\vec{a}|^2=5a^2+5b^2+5c^2-4(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})$

$$=15-4(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})\le15-4\left(\frac{-3}{2}\right)=21$$

$\because\quad \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}\ge\frac{-3}{2}$

12. $$16|\vec{a}||\vec{b}|\sin\frac{\pi}{2}=3|\vec{a}|^2+3|\vec{b}|^2+6|\vec{a}||\vec{b}|$$

$\Rightarrow\quad 3a^2-10ab+3b^2=0\quad\Rightarrow(3a-b)(a-3b)=0$

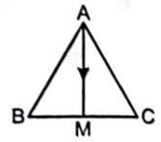
Now $\quad\overrightarrow{OC}\cdot\overrightarrow{AB}=(\vec{a}+\vec{b})\cdot(\vec{b}-\vec{a})=|\overrightarrow{OC}||\overrightarrow{AB}|\cos\theta$

$\Rightarrow\quad \frac{b^2-a^2}{\sqrt{a^2+b^2}\sqrt{a^2+b^2}}=\cos\theta=\frac{9a^2-a^2}{9a^2+a^2}\qquad$ (using $b=3a$)

$\therefore\quad \cos\theta=\frac{4}{5}$

$$\therefore \qquad \tan\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}=\sqrt{\frac{1-\frac{4}{5}}{1+\frac{4}{5}}}=\frac{1}{3}$$

13. $\overrightarrow{AM}=\frac{1}{2}(\overrightarrow{AB}+\overrightarrow{AC})$



14. $\begin{vmatrix} 2 & \lambda & 3 \\ 3 & 3 & 5 \\ \lambda & 2 & 2 \end{vmatrix}=0 \quad \Rightarrow \lambda^2-3\lambda+2=0$

15. $(\vec{a}+\vec{b}-\vec{c})\cdot\left[(\vec{b}+\vec{c}-\vec{a})\times(\vec{c}+\vec{a}-\vec{b})\right]$

$$(\vec{a}+\vec{b}-\vec{c})\cdot(\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}-\vec{c}\times\vec{b}-\vec{a}\times\vec{c}+\vec{a}\times\vec{b})=2(\vec{a}+\vec{b}-\vec{c})\cdot(\vec{b}\times\vec{c}+\vec{c}\times\vec{a})$$

$$=2([\vec{a}\ \vec{b}\ \vec{c}]+[\vec{b}\ \vec{c}\ \vec{a}])=4[\vec{a}\ \vec{b}\ \vec{c}]$$

16. $(\hat{a}\times\hat{b})\times(\hat{a}+\hat{b})=(\hat{a}\cdot(\hat{a}+\hat{b}))\,\hat{b}-(\hat{b}\cdot(\hat{a}+\hat{b}))\,\hat{a}=(1+\hat{a}\cdot\hat{b})(\hat{b}-\hat{a})$

17. Angle between planes is angle between $\vec{n}_1$ and $\vec{n}_2$, where $\vec{n}_1=\overrightarrow{AB}\times\overrightarrow{AC}$ and $\vec{n}_2=\overrightarrow{AD}\times\overrightarrow{AC}$

$$\vec{n}_1=-2\hat{i}+4\hat{j}-3\hat{k}, \qquad \vec{n}_1=6\hat{i}+3\hat{j}-6\hat{k}$$

18. $\vec{a}_1=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$, $\vec{a}_2=x_2\hat{i}+y_2\hat{j}+z_2\hat{k}$ and $\vec{a}_3=x_3\hat{i}+y_3\hat{j}+z_3\hat{k}$ are mutually perpendicular unit vectors, then

$$[\vec{a}_1\ \vec{a}_2\ \vec{a}_3]=\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}=\pm 1$$

22. On solving, $Ax=C$ and $Bx=D$

$$x=\begin{bmatrix}1\\2\\3\end{bmatrix} \qquad x=\begin{bmatrix}3\\1\\2\end{bmatrix}$$

$$P=(1,2,3),\ Q=(3,1,2)$$

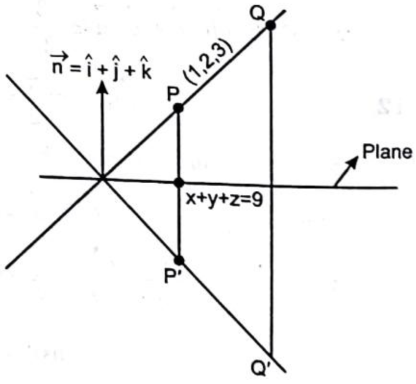
$$PP':\ \frac{x-1}{1}=\frac{y-2}{1}=\frac{z-3}{1}=\lambda$$

$(\lambda+1,\lambda+2,\lambda+3)$ lies on plane

$$3\lambda+6=9 \Rightarrow \lambda=1$$

$\therefore \qquad P'=(3,4,5)$

Similarly $\quad Q'=(5,3,4)$



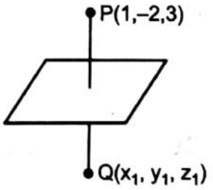
Now check the options.

23. $\overrightarrow{AM} = (\alpha - 1)\hat{i} + \hat{j}$

$\overrightarrow{BM} = (\alpha - 1)\hat{i}$

$\overrightarrow{CM} = (\alpha - 3)\hat{i} + 2\hat{j} + 2\hat{k}$ are coplanar, then $\begin{vmatrix} \alpha - 1 & 1 & 0 \\ \alpha - 2 & 0 & 0 \\ \alpha - 3 & 2 & 2 \end{vmatrix} = 0$

24. Normal vector is parallel to $\overrightarrow{PQ}$



$$\frac{x_1 - 1}{1} = \frac{y_1 + 2}{-1} = \frac{z_1 - 3}{1} = \lambda$$

$\Rightarrow \quad x_1 = \lambda + 1,\ y_1 = -2 - \lambda,\ z_1 = 3 + \lambda$

Mid point of PQ is lie on the plane

$\Rightarrow \quad \lambda = \dfrac{2}{3}$

$$Q\left(\frac{5}{3}, \frac{-8}{3}, \frac{11}{3}\right)$$

25. $$|\hat{a} - \hat{b}| = 1$$

$\Rightarrow \quad \cos\theta = \dfrac{1}{2}$

Volume of parallelopiped $= [\hat{a}\ \ \hat{b}\ \ \hat{a}\times\hat{b}] = \sin^2\theta = \dfrac{3}{4}$

26. Equation of line PQ

$$\frac{x - 3}{1} = \frac{y - 7}{2} = \frac{z - 1}{-6} = \lambda$$

Point $Q(3 + \lambda, 7 + 2\lambda, 1 - 6\lambda)$

If it lies on plane $3x + 2y + 11z = 9$, then

$$\lambda = \frac{25}{59}$$

27. $V_1 = [\vec{a}\ \ \vec{b}\ \ \vec{c}]$

$V_2 = [\vec{a} + \vec{b} - 2\vec{c}\ \ \ 3\vec{a} - 2\vec{b} + \vec{c}\ \ \ \vec{a} - 4\vec{b} + 2\vec{c}] = 15[\vec{a}\ \ \vec{b}\ \ \vec{c}]$

28. Line represented by $x + ay - b = 0,\ cy + z - d = 0$ is parallel to

$$(\hat{i} + a\hat{j}) \times (c\hat{j} + \hat{k}) = a\hat{i} - \hat{j} + c\hat{k}$$

Line represented by $-x + a'y + b' = 0,\ c'y - z + d' = 0$ is parallel to

$$(\hat{i} - a'\hat{j}) \times (c'\hat{j} - \hat{k}) = a'\hat{i} + \hat{j} + c'\hat{k}$$

If these two lines are perpendicular, then

$$aa' + cc' = 1$$

29. Equation of line PQ

$$\vec{r} = (2\hat{i} - 2\hat{j} + 3\hat{k}) + \mu(\hat{i} + 5\hat{j} + \hat{k})$$

$\Rightarrow$ Co-ordinate of $Q(2 + \mu, 5\mu - 2, 3 + \mu)$

If point Q lies on plane, then

$$\mu = \frac{10}{27}$$

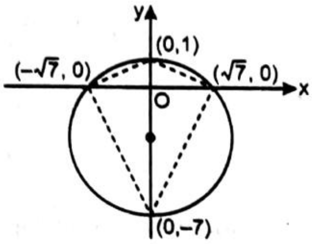
$$\overrightarrow{PQ} = \mu\hat{i} + 5\mu\hat{j} + \mu\hat{k} = \frac{10}{27}\hat{i} + \frac{50}{27}\hat{j} + \frac{10}{27}\hat{k}$$

30. $$(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$$

$$(\vec{a} \cdot \vec{c})\,\vec{b} - (\vec{b} \cdot \vec{c})\,\vec{a} = (\vec{a} \cdot \vec{c})\,\vec{b} - (\vec{a} \cdot \vec{b})\,\vec{c}$$

$$\Rightarrow \quad (\vec{a} \cdot \vec{b})\,\vec{c} = (\vec{b} \cdot \vec{c})\,\vec{a}$$

31. Let $$\vec{r} = x\hat{i} + y\hat{j}$$

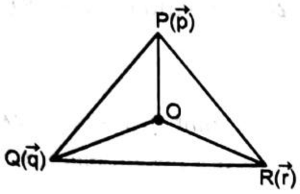


$$\vec{r} \cdot (\vec{r} + 6\hat{i}) = 7$$

$$\Rightarrow \quad x^2 + (y + 3)^2 = 16$$

Area of quadrilateral $= 8\sqrt{7}$

33. $$\frac{1}{2}\frac{|(\vec{p} - \vec{q}) \times (\vec{r} - \vec{q})|}{\frac{1}{2}|\vec{q} \times \vec{r}|} = 4$$



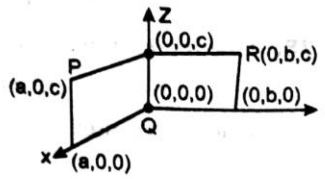
Also, $$\vec{p} + k_1\vec{q} + k_2\vec{r} = 0$$

$$\Rightarrow \quad \vec{p} = -k_1\vec{q} - k_2\vec{r} = 0$$

$$\Rightarrow \quad k_1 + k_2 + 1 = 4$$

$$\Rightarrow \quad k_1 + k_2 = 3$$

34. Let length, breadth and height of rectangular box be a, b, c respectively.



$$\vec{P} = a\hat{i} + c\hat{k}$$

$$\vec{R} = b\hat{j} + c\hat{k}$$

$$\vec{O} = \frac{a}{2}\hat{i} + \frac{b}{2}\hat{j} + \frac{c}{2}\hat{k}$$

$$|\overrightarrow{OQ}||\overrightarrow{OR}|\cos\theta = \left(\frac{a}{2}\hat{i} + \frac{b}{2}\hat{j} + \frac{c}{2}\hat{k}\right) \cdot \left(\frac{a}{2}\hat{i} - \frac{b}{2}\hat{j} - \frac{c}{2}\hat{k}\right)$$

$\Rightarrow \qquad \cos\theta = -\dfrac{1}{3}$

Similarly, $\qquad \cos\phi = -\dfrac{1}{3}$

36. $\vec{\mathbf{r}} = a(\vec{\mathbf{m}} \times \vec{\mathbf{n}}) + b(\vec{\mathbf{n}} \times \vec{\mathbf{l}}) + c(\vec{\mathbf{l}} \times \vec{\mathbf{m}})$

where $\quad [\vec{\mathbf{l}}\,\vec{\mathbf{m}}\,\vec{\mathbf{n}}] = 4, \quad \vec{\mathbf{r}} \cdot \vec{\mathbf{l}} = 4a, \quad \vec{\mathbf{r}} \cdot \vec{\mathbf{m}} = 4b, \quad \vec{\mathbf{r}} \cdot \vec{\mathbf{n}} = 4c$

which imply that

$$\frac{a+b+c}{\vec{\mathbf{r}} \cdot (\vec{\mathbf{l}} + \vec{\mathbf{m}} + \vec{\mathbf{n}})} = \frac{1}{4}$$

37. The volume tetrahedron is given by $k = \dfrac{1}{6}[\vec{\mathbf{a}}\,\vec{\mathbf{b}}\,\vec{\mathbf{c}}] \Rightarrow [\vec{\mathbf{a}}\,\vec{\mathbf{b}}\,\vec{\mathbf{c}}] = 6k$

The volume of parallelepiped is given by

$$\begin{aligned}
[\vec{\mathbf{a}} - \vec{\mathbf{b}} \;\; \vec{\mathbf{b}} + 2\vec{\mathbf{c}} \;\; 3\vec{\mathbf{a}} - \vec{\mathbf{c}}] &= [\vec{\mathbf{a}}\;\vec{\mathbf{b}} + 2\vec{\mathbf{c}}\;3\vec{\mathbf{a}} - \vec{\mathbf{c}}] + [-\vec{\mathbf{b}}\;\vec{\mathbf{b}} + 2\vec{\mathbf{c}}\;3\vec{\mathbf{a}} - \vec{\mathbf{c}}]\\
&= [\vec{\mathbf{a}}\,\vec{\mathbf{b}}\,3\vec{\mathbf{a}} - \vec{\mathbf{c}}] + [\vec{\mathbf{a}}\,2\vec{\mathbf{c}}\,3\vec{\mathbf{a}} - \vec{\mathbf{c}}] + [-\vec{\mathbf{b}}\,\vec{\mathbf{b}}\,3\vec{\mathbf{a}} - \vec{\mathbf{c}}] + [-\vec{\mathbf{b}}\,2\vec{\mathbf{c}}\,3\vec{\mathbf{a}} - \vec{\mathbf{c}}]\\
&= [\vec{\mathbf{a}}\,\vec{\mathbf{b}} - \vec{\mathbf{c}}] + [-\vec{\mathbf{b}}\,2\vec{\mathbf{c}}\,3\vec{\mathbf{a}}] = -[\vec{\mathbf{a}}\,\vec{\mathbf{b}}\,\vec{\mathbf{c}}] - 6[\vec{\mathbf{a}}\,\vec{\mathbf{b}}\,\vec{\mathbf{c}}]\\
&= -7[\vec{\mathbf{a}}\,\vec{\mathbf{b}}\,\vec{\mathbf{c}}]
\end{aligned}$$

Volume is 42 k.

38. We know that the equation of the plane passing through the line of intersection of planes $p_1 = 0$ and $p_2 = 0$ is

$$p_1 + \lambda p_2 = 0$$

That is,

$$(x + 2y + z - 10) + \lambda(3x + y - z - 5) = 0 \qquad \ldots(1)$$

Since, this plane passes through the origin $(0, 0, 0)$ satisfies this equation. This implies that

$$(-10) + \lambda(-5) = 0$$

$\Rightarrow \qquad \lambda = -2$

Substituting the value of λ in Eq. (1), we get

$$(x + 2y + z - 10) - 2(3x + y - z - 5) = 0$$

That is, $\qquad -5x + 3z = 0$

$\Rightarrow \qquad 5x - 3z = 0$

39. Let the point $P(x_p, y_p, z_p)$ be the required point.

The distance of the point from x-axis is $\sqrt{y_p^2 + z_p^2}$.

The distance from the point $(1, -1, 2)$ is

$$\sqrt{(x_p-1)^2+(y_p+1)^2+(z_p-2)^2}$$

$$\Rightarrow \quad y_p^2+z_p^2=(x_p-1)^2+(y_p+1)^2+(z_p-2)^2$$

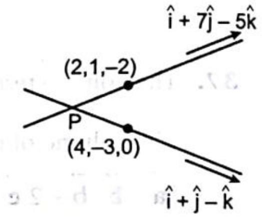
$$\Rightarrow \quad x_p^2-2x_p+2y_p-4z_p+6=0$$

Therefore, the locus of point P is

$$x^2-2x+2y-4z+6=0$$

Exercise-2 : One or More than One Answer is/are Correct

3. Point P on line L_1



$$P(2+\lambda, 1+7\lambda, -2-5\lambda)$$

Point P on line L_2

$$P(4+r, -3+r, -r) \Rightarrow \lambda=-1,\ r=-3$$

Acute angle between L_1 and L_2

$$\cos\theta=\frac{13}{15}$$

Equation of plane containing L_1 and L_2 is $x+2y+3z+2=0$

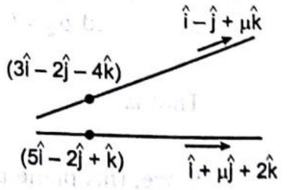
4. $\hat{a}=\hat{b}+(\hat{b}\times\hat{c})$

$\hat{a}\cdot\hat{b}=1$ and $\hat{a}\cdot\hat{c}=\hat{b}\cdot\hat{c}$

$|\hat{a}-\hat{b}|=|\hat{b}\times\hat{c}| \Rightarrow \sin\theta=0 \quad (\because \theta=\hat{b}\hat{c})$

$|\hat{a}+\hat{b}+\hat{c}|^2=3+2(\hat{a}\cdot\hat{b}+\hat{b}\cdot\hat{c}+\hat{a}\cdot\hat{c})=5+4(\hat{b}\cdot\hat{c})$

5. If these two lines are coplanar, then



$\hat{i}-\hat{j}+\mu\hat{k}$
$(3\hat{i}-2\hat{j}-4\hat{k})$
$(5\hat{i}-2\hat{j}+\hat{k})$
$\hat{i}+\mu\hat{j}+2\hat{k}$

$$\begin{vmatrix} 1 & -1 & \mu \\ 1 & \mu & 2 \\ 2 & 0 & 5 \end{vmatrix}=0$$

$$\Rightarrow \quad 2\mu^2-5\mu-1=0$$

6. $\hat{i}\times[(\vec{a}-\hat{j})\times\hat{i}]+\hat{j}\times[(\vec{a}-\hat{k})\times\hat{j}]+\hat{k}\times[(\vec{a}-\hat{i})\times\hat{k}]=0$

$$2\vec{a}-(\hat{i}+\hat{j}+\hat{k})=0 \quad \Rightarrow (2x-1)\hat{i}+(2y-1)\hat{j}+(2z-1)\hat{k}=0$$

$$\Rightarrow x=y=z=\frac{1}{2}$$

7. $[\vec{a}\times\vec{b}\quad \vec{c}\times\vec{d}\quad \vec{e}\times\vec{f}]=(\vec{a}\times\vec{b})\cdot[(\vec{c}\times\vec{d})\times(\vec{e}\times\vec{f})]=(\vec{c}\times\vec{d})\cdot[(\vec{e}\times\vec{f})\times(\vec{a}\times\vec{b})]$

$$=(\vec{e}\times\vec{f})\cdot[(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})]$$

$$=(\vec{a}\times\vec{b})\cdot[(\vec{c}\times\vec{d})\cdot\vec{f}]\,\vec{e}-[(\vec{c}\times\vec{d}\cdot\vec{e})\times\vec{f}]$$

$$=[\vec{c}\ \vec{d}\ \vec{f}][\vec{a}\ \vec{b}\ \vec{e}]-[\vec{c}\ \vec{d}\ \vec{e}][\vec{a}\ \vec{b}\ \vec{f}]$$

Similarly, solve other 2.

8. $3(\vec{a}-\vec{b})+(\vec{b}-\vec{c})+2(\vec{c}-\vec{d})=0$

$$\frac{\overrightarrow{BC}+2\,\overrightarrow{CD}}{1+2}=\overrightarrow{BA}$$

10. $\vec{b}=2\hat{c}+\lambda\hat{a}$

$$|\vec{b}|^2=4+\lambda^2+4\lambda\left(\frac{1}{4}\right)=16 \Rightarrow \lambda=-4,3$$

11. $L_1: x=y=z$

$L_2: \dfrac{x-1}{1}=\dfrac{y+1}{-1}=\dfrac{z}{-1}$

Shortest distance $=\dfrac{1}{\sqrt{2}}$

Equation of plane containing line L_2 and parallel to L_1

$$y-z+1=0$$

Distance of origin from this plane $=\dfrac{1}{\sqrt{2}}$

12. $\vec{r}\cdot(\vec{a}+\vec{b}+\vec{c})=0$

$\Rightarrow\ [\vec{a}\,\vec{b}\,\vec{c}](\sin x+\cos y+2)=0$

$\Rightarrow\ \sin x=-1$ and $\cos y=-1$

13. $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=[(\vec{a}\times\vec{b})\cdot\vec{d}]\,\vec{c}-(\vec{a}\times\vec{b}\cdot\vec{c})\,\vec{d}=r\,\vec{c}+s\,\vec{d}$

where $r=[\vec{a}\ \vec{b}\ \vec{c}]$ and $s=-[\vec{a}\ \vec{b}\ \vec{c}]$ as $\vec{c}$ and $\vec{d}$ are non-collinear.

Similarly, $h=-[\vec{b}\ \vec{c}\ \vec{d}]$ and $k=[\vec{a}\ \vec{c}\ \vec{d}]$

14. Here, $\vec{\alpha}=\hat{i}+2\hat{j},\ \vec{\beta}=2\hat{i}+a\hat{j}+10\hat{k}$ and $\vec{\gamma}=12\hat{i}+20\hat{j}+a\hat{k}$

$$\therefore\ [\vec{\alpha}\,\vec{\beta}\,\vec{\gamma}]=\begin{vmatrix}1 & 2 & 0\\ 2 & a & 10\\ 12 & 20 & a\end{vmatrix}=a^2-24a+240>0, \text{for all } a$$

$\therefore\ \vec{\alpha},\vec{\beta}$ and $\vec{\gamma}$ are non-coplanar or linearly independent for all a.

Hence, (a, b, c) is the correct answer.

19. Let $\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

If $\vec{\mathbf{r}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} + \hat{\mathbf{k}} \quad \Rightarrow \quad -y\hat{\mathbf{k}} + z\hat{\mathbf{j}} = \hat{\mathbf{j}} + \hat{\mathbf{k}}$

$\Rightarrow \quad \vec{\mathbf{r}} = x\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$

If $\vec{\mathbf{r}} \times \hat{\mathbf{j}} = \hat{\mathbf{i}} + \hat{\mathbf{k}} \quad \Rightarrow \quad x\hat{\mathbf{k}} - z\hat{\mathbf{i}} = \hat{\mathbf{i}} + \hat{\mathbf{k}}$

$\vec{\mathbf{r}} = \hat{\mathbf{i}} + y\hat{\mathbf{j}} - \hat{\mathbf{k}}$

20. (A) See dot product

(C) $y = \ln(e^{-2} + e^{x})$

$e^{y} - e^{-2} = e^{x}$

21. $(-3 - 4\lambda, 6 + 3\lambda, 2\lambda) = (-2 - 4\mu, 7 + \mu, \mu)$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 3

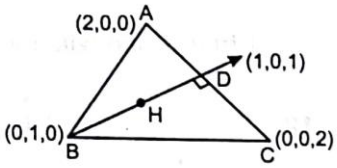
1. $AB = BC$

p.v. of $H = \hat{j} + r(\hat{i} - \hat{j} + \hat{k})$

Also, $\quad \overrightarrow{AH} \cdot \overrightarrow{BC} = 0$

$\Rightarrow \quad [(r-2)\hat{i} + (1-r)\hat{j} + r\hat{k}] \cdot (-\hat{j} + 2\hat{k}) = 0$

$\Rightarrow \quad r - 1 + 2r = 0 \quad \Rightarrow \quad r = \frac{1}{3}$



2. p.v. of $H = \frac{\hat{i}}{3} + \frac{2\hat{j}}{3} + \frac{\hat{k}}{3}$

p.v. of centroid $= \frac{2}{3}\hat{i} + \frac{\hat{j}}{3} + \frac{2\hat{k}}{3}$

p.v. of $S = \frac{3(\text{p.v.}) \text{ of centroid} - \text{p.v. of } \vec{H}}{2}$

y coordinate of $S = \frac{1}{6}$

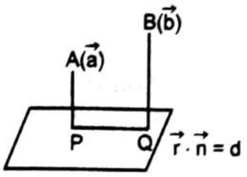
3. Let $P \equiv (a, b, c)$

$\Rightarrow \quad (a-2)^2 + b^2 + c^2 = a^2 + (b-1)^2 + c^2 = a^2 + b^2 + (c-2)^2 = a^2 + b^2 + c^2$

$\Rightarrow \quad P = \left(1, \frac{1}{2}, 1\right)$

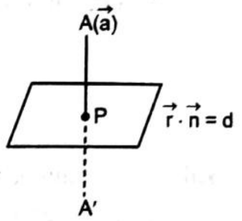
$PA = \frac{3}{2}$

Paragraph for Question Nos. 4 to 6



4. $PQ=(\vec{b}-\vec{a})\cos\theta$ (where θ angle between AB and plane)

$$=\frac{|(\vec{b}-\vec{a})\times\vec{n}|}{|\vec{n}|}$$



5. Equation of line AP is $\vec{r}=\vec{a}+\lambda\vec{n}$

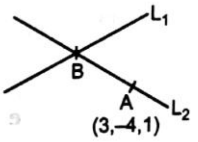
For point $P(\vec{a}+\lambda\vec{n})\cdot\vec{n}=d$

$$\lambda=\frac{d-\vec{a}\cdot\vec{n}}{|\vec{n}|^2}$$

$$\therefore\quad P\left(\vec{a}+\frac{d-\vec{a}\cdot\vec{n}}{|\vec{n}|^2}\right)\quad\therefore A'\text{ is }\vec{a}+2\left(\frac{d-\vec{a}\cdot\vec{n}}{|\vec{n}|^2}\vec{n}\right)$$

6. Distance $=|BQ-AP|=\left|\frac{\vec{b}\cdot\vec{n}-d}{|\vec{n}|}-\frac{\vec{a}\cdot\vec{n}-d}{|\vec{n}|}\right|=\left|\frac{(\vec{b}-\vec{n})\cdot\vec{n}}{|\vec{n}|}\right|$

Paragraph for Question Nos. 7 to 9



7. $B(3+2\lambda,-1-3\lambda,2-\lambda)$

d_r of $L_2<2\lambda,-3\lambda+3,1-\lambda>$

L_2 is parallel to plane $\vec{r}\cdot(2\hat{i}+\hat{j}-\hat{k})=5$

$$\therefore\quad 4\lambda-3\lambda+3-1+\lambda=0$$
$$2\lambda=-2\ \Rightarrow\ \lambda=-1$$

$\therefore\quad B(1,2,3)$

So, equation of L_2 is $\vec{r}=(3\hat{i}-4\hat{j}+\hat{k})+\lambda(\hat{i}-3\hat{j}-\hat{k})$

8. Equation of plane contain L_1 & L_2 is

$$\begin{vmatrix} x-3 & y+1 & z-2 \\ 0 & 3 & 1 \\ -1 & 6 & 2 \end{vmatrix}=0$$

i.e., $(x-3)(6-6)+(y+1)(0+1)+(z-2)(0+3)=0$

$$y+3z-5=0$$

9. Any point of L_1 is $(3+2\lambda,-1-3\lambda,2-\lambda)$

if on plane π, then

$$2(3+2\lambda)+1(-1-3\lambda)-1(2-\lambda)=5$$

$$2\lambda=2 \Rightarrow \lambda=1$$

$\therefore \quad Q(5,-4,1)$

if on xy plane, then $\quad 2-\lambda=0 \Rightarrow \lambda=2$

$\therefore \quad R(7,-7,0)$

$$\text{Volume of tetrahedran} = \frac{1}{6}[\overrightarrow{OA} \quad \overrightarrow{OQ} \quad \overrightarrow{OR}] = \frac{1}{6}\begin{vmatrix} 3 & -4 & 1 \\ 5 & -4 & 1 \\ 7 & -7 & 0 \end{vmatrix} = \frac{7}{3}$$

Paragraph for Question Nos. 10 to 11

Sol. Use crammer rule,

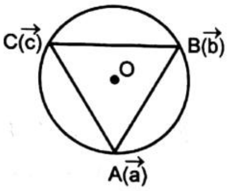
Intersect at a unique point $\Rightarrow D \neq 0$

Do not have any common point of intersection.

$\Rightarrow \quad D=0$ and atleast any one of D_x, D_y, D_z is non-zero (condition of no solution)

Paragraph for Question Nos. 12 to 14

Sol. $|\vec{\mathbf{a}}|=|\vec{\mathbf{b}}|=|\vec{\mathbf{c}}|=r$



$$\text{PV of } E: \frac{\vec{\mathbf{a}}+\left(\dfrac{\vec{\mathbf{a}}+\vec{\mathbf{b}}}{2}\right)+\vec{\mathbf{c}}}{3}$$

$$\vec{\mathbf{e}}=\frac{3\vec{\mathbf{a}}+\vec{\mathbf{b}}+2\vec{\mathbf{c}}}{6}$$

$$\text{PV of } G: \vec{\mathbf{g}}=\frac{\vec{\mathbf{a}}+\vec{\mathbf{b}}+\vec{\mathbf{c}}}{3}$$

12. $\overrightarrow{\mathbf{OE}}\cdot\overrightarrow{\mathbf{CD}}=0 \Rightarrow \left(\dfrac{3\vec{\mathbf{a}}+\vec{\mathbf{b}}+2\vec{\mathbf{c}}}{6}\right)\cdot\left(\dfrac{\vec{\mathbf{a}}+\vec{\mathbf{b}}}{2}-\vec{\mathbf{c}}\right)=0$

$\Rightarrow \quad \vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}-\vec{\mathbf{c}})=0$

$\therefore \quad \overrightarrow{\mathbf{OA}} \perp \overrightarrow{\mathbf{BC}}$

$\therefore \quad \Delta ABC$ must be isosceles with base BC.

$\therefore \quad |\overrightarrow{\mathbf{AC}}|=|\overrightarrow{\mathbf{AB}}|$

13. $\overrightarrow{GE}\cdot\overrightarrow{CD}=0 \Rightarrow \left(\dfrac{3\vec{\mathbf{a}}+\vec{\mathbf{b}}+2\vec{\mathbf{c}}}{6}-\dfrac{\vec{\mathbf{a}}+\vec{\mathbf{b}}+\vec{\mathbf{c}}}{3}\right)\cdot\left(\dfrac{\vec{\mathbf{a}}+\vec{\mathbf{b}}}{2}-\vec{\mathbf{c}}\right)=0$

$\Rightarrow (\vec{\mathbf{a}}-\vec{\mathbf{b}})\cdot\vec{\mathbf{c}}=0 \Rightarrow \overrightarrow{AB}\perp\overrightarrow{OC}$

∴ *ABC* must be isosceles with base *AB*.

∴ Circumcentre and centroid lie on median through C.

∴ Orthocenter also lie on median through C.

14. $[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AB}\times\overrightarrow{AC}]=(\overrightarrow{AB}\ \overrightarrow{AC})^2$

$(\vec{\mathbf{a}}\times\vec{\mathbf{b}}+\vec{\mathbf{b}}\times\vec{\mathbf{c}}+\vec{\mathbf{c}}\times\vec{\mathbf{a}})^2$

$$[\overrightarrow{AE}\ \overrightarrow{AG}\ \overrightarrow{AE}\times\overrightarrow{AG}]=(\overrightarrow{AE}\times\overrightarrow{AG})^2=\left\{\frac{-1}{18}(\vec{\mathbf{a}}\times\vec{\mathbf{b}}+\vec{\mathbf{b}}\times\vec{\mathbf{c}}+\vec{\mathbf{c}}\times\vec{\mathbf{a}})\right\}^2$$

$$=\frac{1}{324}(\overrightarrow{AB}\times\overrightarrow{AC})^2$$

Paragraph for Question Nos. 15 to 16

15. $D(3,-1,2)$ *AB* lies along $(0,1,2)$

CD lies along $(3,-2,0)$

Equation of plane containing *AB* line

$$\begin{vmatrix} x-1 & y-1 & z-1 \\ 0 & 1 & 2 \\ 2 & -2 & 0 \end{vmatrix}=2(x-1)+2(y-1)-(z-1)=0$$

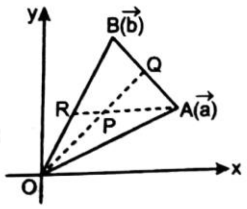
Containing *CD* line $2(x-1)+2(y-1)-(z-2)=0$

16. $r=(3,-1,2)+d(1,0,0)$

Equation of *ABC* plane is $x=1$.

Paragraph for Question Nos. 17 to 18

17. $R\left(\dfrac{2\vec{\mathbf{b}}}{5}\right)$ and $Q\left(\dfrac{3\vec{\mathbf{b}}+2\vec{\mathbf{a}}}{5}\right)$



$$\frac{\mu\left(\dfrac{3\vec{\mathbf{b}}+2\vec{\mathbf{a}}}{5}\right)}{\mu+1}=\frac{\lambda\vec{\mathbf{a}}+1\left(\dfrac{2\vec{\mathbf{b}}}{5}\right)}{\lambda+1}$$

$$\Rightarrow \frac{2\mu}{5(\mu+1)}=\frac{\lambda}{\lambda+1} \text{ and } \frac{3\mu}{5(\mu+1)}=\frac{2}{5(\lambda+1)} \Rightarrow \mu=\frac{10}{9}$$

18. $\text{Ar}(\Delta OPA) = \frac{1}{2}\left|\overrightarrow{\mathbf{OP}} \times \overrightarrow{\mathbf{OA}}\right| = \frac{1}{2}\left[\frac{2}{19}(3\vec{\mathbf{b}} + 2\vec{\mathbf{a}}) \times \vec{\mathbf{a}}\right] = \frac{3}{19}(\vec{\mathbf{b}} \times \vec{\mathbf{a}})$

$$\text{Ar}(PQBR) = \frac{1}{2}\left|\overrightarrow{\mathbf{OQ}} \times \overrightarrow{\mathbf{OB}} - \overrightarrow{\mathbf{OP}} \times \overrightarrow{\mathbf{OR}}\right| = \frac{1}{2}\left[\left(\frac{3\vec{\mathbf{b}} + 2\vec{\mathbf{a}}}{5}\right) \times \vec{\mathbf{b}} - \frac{2}{19}(3\vec{\mathbf{b}} + 2\vec{\mathbf{a}}) \times \frac{2\vec{\mathbf{b}}}{5}\right]$$

$$= \frac{3}{19}(\vec{\mathbf{a}} \times \vec{\mathbf{b}})$$

Exercise-4 : Matching Type Problems

1. (A) Line $\frac{x-1}{-2} = \frac{y+2}{3} = \frac{z}{-1}$ is along the vector $\vec{a} = -2\hat{i} + 3\hat{j} - \hat{k}$ and line $\vec{r} = (3\hat{i} - \hat{j} + \hat{k}) + t(\hat{i} + \hat{j} + \hat{k})$ is along the vector $\vec{b} = \hat{i} + \hat{j} + \hat{k}$. Here $\vec{a} \perp \vec{b}$.

Also, $$\begin{vmatrix} 3-1 & -1-(-2) & 1-0 \\ -2 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix} \neq 0$$

(B) The direction ratios of the line $x - y + 3z - 4 = 0 = 2x + y - 3z + 5 = 0$ are

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{vmatrix} = \hat{i} + 7\hat{j} + 3\hat{k}$$

Hence, the give two lines are parallel.

(C) The given lines are

$(x = t - 3, y = 2t + 1, z = -3t - 2)$ and $\vec{r} = (t+1)\hat{i} + (2t+3)\hat{j} + (-t-9)\hat{k}$,

or $$\frac{x+3}{1} = \frac{y-1}{-2} = \frac{z+2}{-3} \text{ and } \frac{x-1}{1} = \frac{y-3}{2} = \frac{z+9}{-1}$$

The lines are perpendicular as $(1)(1) + (-2)(2) + (-3)(-1) = 0$

Also, $$\begin{vmatrix} -3-1 & 1-3 & -2-(-9) \\ 1 & -2 & -3 \\ 1 & 2 & -1 \end{vmatrix} = 0$$

Hence, the lines are intersecting.

(D) The given lines are $\vec{r} = (\hat{i} + 3\hat{j} - \hat{k}) + t(2\hat{i} - \hat{j} - \hat{k})$ and $\vec{r} = (-\hat{i} - 2\hat{j} + 5\hat{k}) + s(\hat{i} - 2\hat{j} + \frac{3}{4}\hat{k})$.

$$\begin{vmatrix} 1-(-1) & 3-(-2) & -1-5 \\ 2 & -1 & -1 \\ 1 & -2 & 3/4 \end{vmatrix} = 0$$

Hence, the lines are coplanar and hence intersecting (as the lines are not parallel).

2. (A) If $\vec{a}, \vec{b}$ and $\vec{c}$ are mutually perpendicular, then $[\vec{a}\times\vec{b}\quad \vec{b}\times\vec{c}\quad \vec{c}\times\vec{a}] = [\vec{a}\quad \vec{b}\quad \vec{c}]^2$

$$= (|\vec{a}||\vec{b}||\vec{c}|)^2 = 16$$

(B) Given $\vec{a}$ and $\vec{b}$ are two unit vectors, *i.e.*, $|\vec{a}| = |\vec{b}| = 1$ and angle between them is $\frac{\pi}{3}$.

$$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|} \Rightarrow \sin\frac{\pi}{3} = |\vec{a}\times\vec{b}|;\quad \frac{\sqrt{3}}{2} = |\vec{a}\times\vec{b}|$$

Now $[\vec{a}\quad \vec{b}+\vec{a}\times\vec{b}\quad \vec{b}] = [\vec{a}\ \vec{b}\ \vec{b}] + [\vec{a}\ \vec{a}\times\vec{b}\ \vec{b}] = 0 + [\vec{a}\ \vec{a}\times\vec{b}\ \vec{b}]$

$$= (\vec{a}\times\vec{b})\cdot(\vec{b}\times\vec{a}) = -|\vec{a}\times\vec{b}|^2 = -\frac{3}{4}$$

(C) If $\vec{b}$ and $\vec{c}$ are orthogonal, $\vec{b}\cdot\vec{c} = 0$

Also, it is given that $\vec{b}\times\vec{c} = \vec{a}$

Now $[\vec{a}+\vec{b}+\vec{c}\quad \vec{a}+\vec{b}\quad \vec{b}+\vec{c}] = [\vec{a}\quad \vec{a}+\vec{b}\quad \vec{b}+\vec{c}] + [\vec{b}+\vec{c}\quad \vec{a}+\vec{b}\quad \vec{b}+\vec{c}]$

$$= [\vec{a}\quad \vec{b}\quad \vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c}] = \vec{a}\cdot\vec{a} = |\vec{a}|^2 = 1$$

(because $\vec{a}$ is a unit vector)

(D) $[\vec{x}\quad \vec{y}\quad \vec{a}] = 0$

Therefore, $\vec{x}, \vec{y}$ and $\vec{a}$ are coplanar.

$$[\vec{x}, \vec{y}\ \vec{b}] = 0$$

Therefore, $\vec{x}, \vec{y}$ and $\vec{b}$ are coplanar.

Also, $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

Therefore, $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar.

From (i), (ii) and (iii)

$\vec{x}, \vec{y}$ and $\vec{c}$ are coplanar. Therefore, $[\vec{x}, \vec{y}\ \vec{c}] = 0$

Exercise-5 : Subjective Type Problems

1. Line L is the shortest distance line of given lines.

2. $[\hat{a}\ \ \hat{b}\ \ \hat{c}] = [\hat{b}\times\hat{c}\ \ \hat{c}\times\hat{a}\ \ \hat{a}\times\hat{b}] = [\hat{a}\ \ \hat{b}\ \ \hat{c}]^2$

$$\Rightarrow \qquad [\hat{a}\ \ \hat{b}\ \ \hat{c}] = 1$$

Projection of $\hat{b}+\hat{c}$ on $\hat{a}\times\hat{b} = \dfrac{(\hat{b}+\hat{c})\cdot(\hat{a}\times\hat{b})}{|\hat{a}\times\hat{b}|} = \dfrac{[\hat{a}\ \ \hat{b}\ \ \hat{c}]}{|\hat{a}\times\hat{b}|}$

3. Let $l = m = n = \dfrac{1}{\sqrt{2}}$

4. $\overrightarrow{OC} = \alpha^2(\vec{a}+\vec{b})^2 + \beta^2(\vec{a}\times\vec{b})^2 + 2\alpha\beta[\vec{a}\cdot(\vec{a}\times\vec{b}) + \vec{b}\cdot(\vec{a}\times\vec{b})]$

$$\Rightarrow \qquad 1 = \alpha^2\left(1+1+2\cdot1\cdot1\frac{1}{2}\right) + \beta^2\cdot1\cdot1\left(\frac{\sqrt{3}}{2}\right)^2 + 0 \qquad \ldots(1)$$

Also, $\qquad \overrightarrow{OB}\cdot\overrightarrow{OC} = |\overrightarrow{OB}|\cdot|\overrightarrow{OC}|\cos\dfrac{\pi}{3}$

$$\Rightarrow \qquad \alpha\cdot1\cdot1\cdot\frac{1}{2} + \alpha\cdot1 = \frac{1}{2} \Rightarrow \alpha = \frac{1}{3} \qquad \ldots(2)$$

From (1) and (2), $\quad \beta^2 = \dfrac{8}{9}$

5. $\vec{v}_{n+1} - \vec{v}_n = \left(\begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \end{bmatrix}\right)^{n+1} \vec{v}_0$

$$\vec{v}_2 - \vec{v}_1 = \begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \end{bmatrix} \vec{v}_0$$

$$\vec{v}_3 - \vec{v}_2 = \begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \end{bmatrix}^3 \vec{v}_0$$

$$\vec{v}_n - \vec{v}_{n-1} = \begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \end{bmatrix}^n \vec{v}_0$$

Adding all the equations,

$\vec{v}_n - \vec{v}_0 = (A + A^2 + A^3 + \ldots\ldots A^n)\,\vec{v}_0$

where $A = \begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \end{bmatrix} \Rightarrow \vec{v}_n = (I + A + A^2 + \ldots\ldots)\,\vec{v}_0$

6. Let $$B = A - \frac{1}{3}A^2 + \frac{1}{9}A^3 - \frac{1}{27}A^4 + \ldots\ldots$$

$$-\frac{AB}{3} = -\frac{A^2}{3} + \frac{1}{9}A^3 - \frac{1}{27}A^4 + \ldots\ldots$$

$$\left(I + \frac{A}{3}\right)B = A$$

$$B = \frac{1}{3}(3I + A)^{-1}A$$

7. $\det M_n = \sum_{k=0}^{n}\left(\frac{1}{(2k+1)!} - \frac{1}{(2k+2)!}\right) = \frac{1}{1!} - \frac{1}{(2n+2)!}$

8. $$|\vec{a} + \vec{b}| = \sqrt{3}$$

$\Rightarrow$ Squaring both sides

$\Rightarrow$ $$\vec{a} \cdot \vec{b} = \frac{1}{2}$$

$$\vec{c} = \vec{a} + 2\vec{b} - 3\vec{a} \times \vec{b}$$

$\Rightarrow$ $$\vec{a} \cdot \vec{c} = 2 \;\&\; \vec{b} \cdot \vec{c} = \frac{5}{2}$$

$$p = |(\vec{a} \cdot \vec{a})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}|$$

$$p = \sqrt{\left|2\vec{b} - \frac{5}{2}\vec{a}\right|^2}$$

$$p = \frac{\sqrt{21}}{2} \Rightarrow [p] = 2$$

9. $$\vec{r} = (\vec{a} \times \vec{b})\sin x + (\vec{b} \times \vec{c})\cos y + 2(\vec{c} \times \vec{a})$$

$$\vec{r} \cdot \vec{a} = [\vec{b}\ \vec{c}\ \vec{a}]\cos y$$

$$\vec{r} \cdot \vec{b} = 2[\vec{c}\ \vec{a}\ \vec{b}]$$

$$\vec{r} \cdot \vec{c} = \sin x[\vec{a}\ \vec{b}\ \vec{c}]$$

$\Rightarrow$ $\sin x + \cos y + 2 = 0$

$\Rightarrow$ $$\sin x = -1 \quad \text{and} \quad \cos y = -1$$

$$x = -\frac{\pi}{2} \qquad\qquad y = \pi$$

10. New equation of plane : $4x + 7y + 4z + 81 + \lambda(5x + 3y + 10z - 25) = 0$

$$(4 + 5\lambda)4 + (7 + 3\lambda)7 + (4 + 10\lambda)4 = 0$$

$\Rightarrow$ $\lambda = -1$

$\Rightarrow$ Equation of plane : $x - 4y + 6z - 106 = 0$

$$\text{distance} = \frac{106}{\sqrt{53}} = \sqrt{212}$$

18. $\vec{\omega} \times \vec{\mu} = \vec{\nu}$

$\vec{\nu} \cdot (\vec{\omega} \times \vec{\mu}) = \vec{\nu} \cdot \vec{\nu} = 1$

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